Problems in matching finite elements having different dimensionalities

Anatoly V. Perelmuter

SCAD Group, 13, Chokolovsky blrd., room 508, 03186, Kiev, Ukraine
e–mail: aperel@i.com.ua

Vladimir I. Slivker*

Giprostroymost Co., Saint–Petersburg, Russia
e–mail: slivker@gpsm.ru

Abstract

The paper deals with an analysis of some situations, arising in connection with structural members of different dimensions combined in a design model, the members being described mathematically in totally different ways. The situations like this may arise, say, when a design model includes both one-dimensional (bars) and two-dimensional (plates and/or slabs) members.

Keywords: FEA, design models

1. Introduction

It is only a rare case that a whole structure can be represented by elements of the same type (such as plate elements). More frequently the same design model may include bars and shells, and other types of elements.

Table 1. Possible options of combining parts

Dimensionality	Elements		1	2	3	4	5
One-dimensional	Bars	**1**	+	–	–	–	–
Two-dimensional	Slabs	**2**		+	–	–	–
	Plates	**3**			+	–	–
	Shells	**4**				+	–
Three-dimensional	Bulk elements	**5**					+

Table 1 lists possible options of combining parts of a structure that contain elements of different dimensionality. The "–" sign designates dangerous combinations of structural members from the standpoint of correctness of a design model.

As will be shown below, this merging of a variety of different elements in their common design model requires a careful attention, especially in places where the said elements of different dimensionality are joined.

The common rule is: when joining elements of different dimensionality, one should not rely on formal computational tools provided by finite element software. As a rule, one needs to perform a more detailed analysis of joints between finite elements of different dimensionality to keep the design model correct.

Let's discuss some situations of those listed in Table 1 in more detail.

2. Bars + Plates.

Let's take as an example of combined slab and bar elements the analysis of a spatial framing together with a slab foundation.

So, let us consider a discrete design model combining plate/slab finite elements and bars rigidly attached to the slab. The finite element mesh is formed so that the bars of the structure's framework hit the nodes of the meshed slab. If no additional measures are taken, the model described above will provide the consistency of both vertical displacements of the slab/framework (in the perpendicular direction to the plate) and the respective slopes in nodes where the plate and bar elements join one another. Though, bending moments in cross-sections of the columns adjoining the slab, calculated with this model, have nothing to do with the real distribution of internal stresses.

As can be seen, the bending moment in a bar of this model, independent of the mesh spacing, is transferred to the slab as a concentrated moment in a node of the mesh (the moment is concentrated because the bar is one-dimensional). On the other hand, the slab under the concentrated bending moment acquires an infinite slope in the moment's plane at its point of application. More exactly, the expression of the slope includes a logarithmic singularity. Thus the slab does not resist to a concentrated slope and does not restrain the framework elements.

In order to be particular, let's refer to an accurate analysis of a simply supported round plate loaded with an external concentrated moment M at its center. Denoting the plate's radius as R, its flexural rigidity as D, its material's Poisson ratio as ν, the current radial and angular polar coordinates as ρ and φ, we have the following expression for the plate deflection w [1]

$$w = \frac{MR}{8\pi D}\left[\frac{1+\nu}{3+\nu}\,(\rho^3 - \rho) - 2\rho\ln\rho\right]\cos\varphi \,,. \qquad (1)$$

and the slope θ of the normal to the plate's median surface in the plane of the M application (i.e. at $\varphi = 0$) will be

$$\theta = \left.\frac{\partial w}{\partial \rho}\right|_{\varphi=0} = \frac{MR}{8\pi D}\left[\frac{1+\nu}{3+\nu}\,(3\rho^2 - 1) - 2 - 2\ln\rho\right], \qquad (2)$$

hence the said logarithmic singularity (at $\rho \to 0$) that makes the slope angle θ infinite in the point of application of the concentrated moment.

In order to amend the model, one should allow for the design of the joint between the framework and the slab. If the columns of the framework are attached to the slab via plinths, then the latter can be treated as perfectly rigid bodies not changing their sizes whatever modifications are made to the finite element mesh.

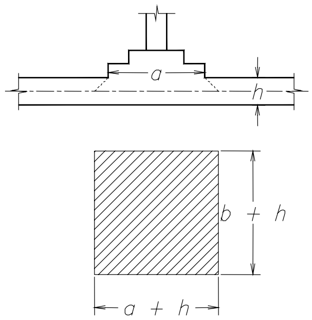

Figure 1: An attachment of a column by a plinth: *a* and *b* are plane sizes of the plinth's foot

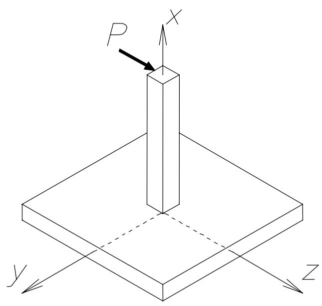

Figure 2: A column fixed at a slab

The size of the rigid body itself can be assigned on the basis of the plinth's size ($a \times b$), allowing for the application of its pressure to the slab at the angle of 45° (Fig. 2), which conforms to the common construction practice.

In this model the concentrated moment from the framework's columns is transferred to the slab via a stiff spacer the sizes of which $a+h$ and $b+h$ do not depend on the spacing of the finite element mesh, nor the discretization error respectively. Though the discretization error will affect the numerical values of the bending moments in the columns, it won't cause the design model to become so fatally different from the real structure as it used to be in the model with no intermediate rigid body. When no plinths are used, one may use the sizes of the column's cross-section as those of the rigid body, taking into account the application of the pressure to the slab at 45 degrees as shown in Fig. 1.

In order to give a confirmation of all said above, let's discuss results of analysis of a square plate clamped along its contour with a single column fixed at the center of the plate. An external concentrated force P is applied to the free top end of the column. It is directed along the global axis Z (Fig. 2). In this case the bending moment in the bottom cross-section of the column is a constant value independent of the finite element mesh spacing because the system is statically determinate with respect to the column. Though, the same effect manifests itself by a horizontal displacement of the column's top that grows infinitely as the mesh is becoming denser because of the growing slope in the central node of the plate.

The second line of Table 2 gives results of calculation of the displacement w_n of the column's free end in the direction of the Z vs. the finite element mesh size (n×n) on a quarter of the plate's plane projection.

The following source data are used by the analysis:

- the plate thickness $h = 0.5$;
- the full plane dimensions of the plate 10.0×10.0;
- the size of the cross-section of the column 0.5×0.5;
- mechanical constants of the column and slab material $E = 3\cdot 10^7$, $\nu = 0.25$.

Table 2. Displacements w_n

n×n	**2×2**	**4×4**	**8×8**	**16×16**	**32×32**	**64×64**
$w_n \cdot 10^4$	11.826	11.996	12.162	12.326	12.492	12.659
$\underline{w}_n \cdot 10^4$	–	11.209	11.194	11.180	11.172	11.169

It is easy to see that each doubling of the mesh causes the deflection wn to grow by almost the same value $\Delta \cong 0.167\cdot 10^{-4}$, or in other words, the column's deflection grows linearly with $\log_2 n$. More exactly,

$$w_n = w_2 + \Delta\cdot\log_2(n-1),$$

wherefrom the infinite growth of the slope of the column's bottom cross-section (as the finite element mesh is becoming denser) follows.

Let's now see how the result of the analysis will change if a perfectly rigid body of a plinth is included therein. Taking the designations from Fig. 5.11, we assume $a + h = b + h = 2.50$.

The third row of Table. 2 presents displacements w_n of the column's free end obtained with the plate model that includes the perfectly rigid body of the sizes indicated above. Even a careless eye looking at these solutions will immediately notice a quick convergence of the solution as the mesh condenses.

Finally, we'd like to draw the reader's attention to the circumstance related to a joint analysis of a spatial framework and its slab foundation. Assuming columns of the framework to be orthogonal, we find that the slab does not resists a column's torsion as well as displacements v, w of the column's bottom in the direction of Y, Z respectively because the plate finite elements just do not have appropriate degrees of freedom to restrain those linear displacements and torsions.

If there are no other external horizontal constraints in the model (those in the plane *Y*, *Z*), then the mechanical system turns out to be unsecured and geometrically unstable, and the software will have to response to this circumstance somehow. An even more tricky bug is the absence of constraints that keep the columns from torsion because in a spatial framework no geometrical instability would arise due to it, and the respective software might lack mechanisms for detecting the errors of this kind. This circumstance should be kept in mind when building a design model of a complex combined structure.

One of feasible solutions to join plate and bar elements can be an imposition of external constraints on the rotations of nodes in the plane of a plate. The engineer must be confident of the impossibility for such rotations to appear.

3. Bars + Plane Stress.

To analyze a high-rise building including both a framework and stiffening diaphragms under horizontal loads, one has to include heterogeneous elements in its design model. Here we are concentrating on a situation when the relative sizes of the stiffening diaphragm do not allow classifying it as a bar element even though the shear be taken into account. In this case the diaphragm is treated as a deep beam, and its behavior is described by the plane elasticity theory.

So, let the design model of a structure include a plate and a framework made of bars, for example, as shown in Fig. 3.

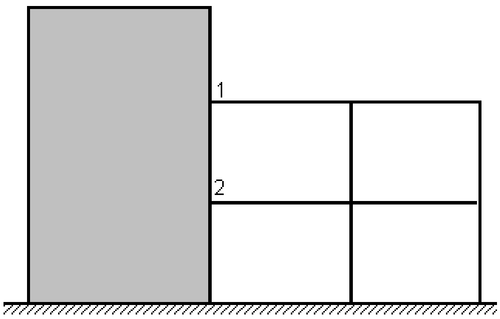

Figure 3: A junction between a framework and a plate

If one takes an utterly formal attitude towards the formation of a discrete design model and describes the plate by a set of plane stress finite elements while the bar framework by one-dimensional elements without any additional measures taken, then both the bending moments and longitudinal/lateral stresses in crossbars that adjoin nodes at the boundary between the framework and the diaphragm (nodes 1 and 2 in Fig. 3) will be totally unrelated to the realities of life.

Let's handle the bending moments first. Let the model of the diaphragm use simplest finite elements with two degrees of freedom in each node (linear displacements u and v in two mutually orthogonal directions). Such finite elements do not resist to a rotation of an adjoining node because the elements just lack the appropriate degrees of freedom. Therefore these finite elements do not transfer any moments to the adjoining nodes. As the software will require all equilibrium conditions to be satisfied, including the equilibrium of each node by moments, the crossbars that join these nodes rigidly will have to transfer zero moments to the nodes. Thus the bending moments in the crossbars in these nodes will be equal to zero. It corresponds to a model of hinged attachment of the crossbars to the diaphragm, and it does not satisfy the requirement of a user oriented at the analysis of crossbars clamped in the diaphragm.

In order to avoid such distortion of the design model, sometimes users introduce external constraints into crossbar-to-diaphragm junction nodes to restrain the latter from rotation. But this technique can hardly be approved because therewith we introduce a distortion to the model again. This time it is an increase of the total stiffness of the model structure because there are no constraints like this in the real structure.

The interesting thing is that in spite of this problem's being known to all the users for a long time, discussions on the subject continue arising again and again. Among recommendations being invented one can hear even an appeal to employ the moment elasticity theory to simulate the diaphragm's behavior. As we will show later, the problem can be solved with simpler models and the behavior of a real structure can be adequately described only by considering structural solutions of joints between the crossbars and the diaphragms and allowing for these solutions in the design model.

Among propagated recommendations, which we cannot uphold for reasons stated below, there is one to use plane stress finite elements having additional degrees of freedom with the meaning of nodal rotations. The finite elements of this particular type are quite feasible and are one of options to build finite element models of higher order of displacement/stress approximation. That's why the text below must not be treated as a negative opinion towards those finite element types. We'd like to draw the reader's attention to the fact that an attempt to circumvent the problem stated above is fruitless, moreover, dangerous because makes a user think this is a solution while actually it is not. Again the gap for this new species of "bugs" is just masked not eliminated (it is the user who may miss the gap, but a "bug" wouldn't!) by the discretization error: it is true that the bending moments in the crossbars are nonzero.

Just as in the problem of the plate-to-bar coupling, the main issue is not a drawback of the discrete model itself. The matter is that the original mathematical formulation of the problem which implies a pointwise joint between a plane stressed diaphragm and a one-dimensional bar, when solved accurately, yields zero moments at the junctions of the bars and the diaphragm. To prove this statement it suffices to make sure that the respective rotation component ωz of the plane elasticity solution is singular in the vicinity of the concentrated moment's application point, that is, tends to infinity.

To see this, let's consider a round plate of the radius *R* restrained from any displacements at its exterior contour as shown in Fig. 4. Let a perfectly rigid disk of the radius *c* be embedded into the plate at its center, and let a torsional moment M be applied to it.

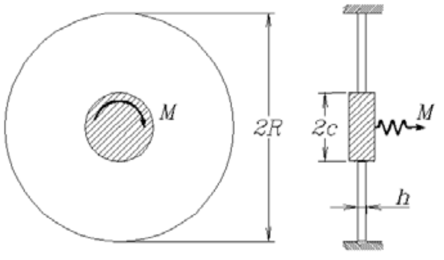

Figure 4: A round plate loaded with a torsion moment

Denoting radial and tangential displacements of the plate's points as u and v, it's easy to see that the components of the displacement vector in any point outside the rigid disk are defined by the expressions

$$u = 0\,, \quad v = \frac{M}{4\pi Gh}\,(1/\rho - \rho/R^2)\,, \tag{3}$$

where ρ is the current radial coordinate, G is the shear modulus, h is the thickness of the plate.

As one can see, all equilibrium equations hold together with the boundary conditions in this case. Denoting the rotation angle (slope) of the rigid disk as ωz, we have the following

$$\omega_z = v(c)\,/\,c = \frac{M}{4\pi Gh}\,(1/c^2 - 1/R^2) \tag{4}$$

wherefrom one can see that the slope ω_z tends to infinity as the size of the disk tends to zero.

Another argument sometimes given "pro" the introduction of additional parameters ω_z is related to the problem of coupling a plane stress plate and a bar along the whole length of the bar rather than pointwise. Such problems arise when one borders plates along their whole or partial boundary by explicit ribs simulated by bar elements. The conventional finite element analysis technique provides the consistence of the bar's and its adjoining plate's displacements in their nodes only, while in intermediate points (between the nodes) one cannot avoid discontinuities of the bar's and plate elements' displacements in the perpendicular direction to the orthogonal bar axis. The introduction of parameters ω^z enables one to get rid of this inconsistence fully or at least partially depending on approximations used to approximate the plate's displacement fields. It's true all right, but let us see what price is paid for this restored consistence of the displacement.

Let's consider a plane elasticity problem posed in an area having a rectilinear fragment of its boundary Γ provided with a bar along the whole length of the boundary.

We introduce a Cartesian coordinate system (X, Y) so that the X axis match the longitudinal axis of the bar and denoting displacements of the bar's cross-sections along the X and Y axes as $u°(x)$ and $v°(x)$ respectively. We should impose the following boundary conditions on the boundary, or, exactly, interface conditions

$$u(x,0) = u°(x)\,, \quad v(x,0) = v°(x)\,, \tag{5}$$

where $u(x,y)$ and $v(x,y)$ are usual components of the displacement vector along the X and Y axis respectively. Also, if we try to identify the bar cross-section slopes $\theta_z = dv°/dx$ with rotation components ω_z at Γ, then we'll have an additional condition to be satisfied at the Γ boundary.

To be particular, let's discuss an introduction of the rotational component ω_z as an averaged slope in the vicinity of a point of the elastic medium, that is

$$\omega_z = (\partial v/\partial x - \partial u/\partial y)/2. \tag{6}$$

In this case the said additional condition at Γ is reduced to a requirement that there be no shear γ_{xy}. By equating the expressions of $\theta_z(x)$ and $\omega_z(x,0)$ we obtain

$$\gamma_{xy}(x,0) = (\partial v/\partial x + \partial u/\partial y)|_{y=0} = 0\,, \tag{7}$$

thus at the Γ boundary there are three conditions instead of two implied by the biharmonic equation. This renders the mathematical formulation of the problem incorrect. It can be easily imagined at the mechanical level, too. Indeed, condition (7) leads to the prohibition of tangential stresses τ_{xy} along the Γ boundary on the basis of physical relations for an isotropic elastic body. It means that the body can slide freely along the Γ boundary. On the other hand, it follows from (5) that all displacement components must acquire certain values at the boundary. Thus the two conditions contradict each other.

The mentioned considerations awaken a suspicion. Therefore the described technique can hardly be recommended for a wide practical employment without a careful (theoretical) study of conditions of its safe application.

We should note it is not our intention to criticize any particular way of introducing an additional degree of freedom into a plane stressed plate. It is not the method of the ω_z parameter introduction which is vicious, it is the very principle of its identification with the θ_z parameter. Whatever sophisticated technique were applied to introduce ω_z into a design model, there would be no reason at all to state the equality

$$\omega_z = \theta_z \tag{?}$$

So, let us consider the finding of a stress and strain distribution in a plate that occupies an area Ω with a boundary Γ in the (X,Y) plane. Let the plate be bordered by a Timoshenko bar along a part of its boundary Γ_1 which we assume to run along the X axis only. We emphasize the use of a Timoshenko bar rather than a Bernoulli one. It is the case where contradictions caused by equality (?) reveal themselves most evidently.

Let the plate is restrained from all displacements at the other part of its boundary $\Gamma_2 = \Gamma - \Gamma_1$. We assume the x coordinate of the longitudinal axis of the bordering bar varies between 0 and l. It is easy to see that the variational formulation of the problem can be reduced to the minimization of the Lagrangian L. Here we don't neglect to present it in its complete form, with a conventional notation:

$$\mathsf{L} = \frac{E}{2\left(1-\nu^2\right)} \int_\Omega \left[u_{,x}^2 + v_{,x}^2 + v_{,x}^2 + 2\nu u_{,x} v_{,x} + \frac{1-\nu}{2}\left(u_{,x} + v_{,x}\right)^2 \right] d\Omega +$$

$$+ \frac{1}{2} \int_0^l [EI\,\theta_{z,x}^2 + GF(\,v_{,x}^{\circ} - \theta_z\,)^2 + EA(\,u_{,x}^{\circ}\,)^2]dx - \Pi,$$

where the first integral represents a deformation energy accumulated in the plate itself, the second one the energy accumulated in the bar, and Π is a drop of potential of external forces. Apparently, the Lagrangian L is a functional of three independently varied functions

$$\mathsf{L} = \mathsf{L}(u,v,\theta_z) \ ,$$

as $v°(x) = v(x,0)$ and $u°(x) = u(x,0)$, so $u°(x)$ and $v°(x)$ are not independently varied functions. The minimum of the functional L is sought in the set of functions $u(x,y)$, $v(x,y)$, $\theta_z(x)$ that satisfy main (kinematical) boundary conditions

$$u = 0 \ , v = 0 \ \in \Gamma_2,$$

and have generalized square-summable first derivatives. That's all required by the variational principle of the full potential energy minimum. The Lagrange variational principle does not establish any predefined relation between the bar section slopes $\theta_z(x)$ and the $u(x,y)$ and $v(x,y)$ displacements of the plate. If now we introduce an expression for the "slope" ω_z in a point (x, y) of the plate, it will have a form like $\omega_z(x,y) = \mathrm{A}(u, v)$ anyway, where the A operator will define the method of this introduction. Now it's clear that the relationship (?) is just an imposition of an additional constraint on the system, the constraint being

$$\mathrm{A}(u, v)\Big|_{y=0} = \theta_z(x),$$

that introduces by no means justified distortion into the original statement of the problem whatever form the A operator might have.

It is apparent that the independence of the θ_z function on functions u and v is lost owing to the condition (?). By the way, this distortion makes the system stiffer, just as does any additional constraint. It implies that the distorted finite element solution is further by the energetic norm from the accurate solution of the problem than the similar finite element solution without the additional constraint (?). In other words, we have the following inequality

$$\mathsf{E} \geq \mathsf{E}_h \geq \mathsf{E}_h^+ \ ,$$

where E is the energy of the system corresponding to the accurate solution, E_h is that of the usual finite element solution, E_h^+ is the energy of the finite element solution disturbed by the additional constraint (?). Apparently, these estimates hold only for fully consistent finite elements. Also, these estimates imply that the external influence on the system consists of forces only.

Now let's discuss longitudinal and lateral stresses in bar elements adjoining a plate. It can be proved that the action of a concentrated force on a plane stressed/strained body causes the displacement of this force's application point to be singular.

For example, consider the same problem of a round plate restrained from all displacements along its contour (Fig. 4) but subjected to a force P applied to the disk and directed along a radius. The respective u displacement of the disk will be

$$u = \frac{P}{8\pi Gh}\left[(3-\nu)\ln(c/R) - \frac{(1+\nu)^2(c^2/R^2-1)}{(3-\nu)(c^2/R^2+1)}\right], \qquad (8)$$

thus there is a logarithmic singularity at $c \to 0$.

Using the same logic with a bending moment, we come to a conclusion that the accurate solution of the pointwise bar-to-plane coupling problem (i.e. the matching in a single node) will yield a zero force transferred to the bar. Again, nonzero values of the longitudinal and lateral stresses in bars formally obtained using a discrete model are caused by the discretization error solely. The design model itself (not to be confused with its discrete counterpart) yields both zero moments and stresses in the bars.

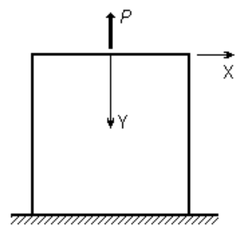

Figure 5: Square plate

The same conclusion can be made after a careful analysis of numerical experiments on the condensation of the mesh in the case of a concentrated force applied to the plate. To do it, let's return to the problem shown in Fig.5.

Table 3 presents results of the computation where vn is the displacement of the P force application point in the direction of Y. It's clear that the displacement tends to infinity as the finite element mesh condenses, exactly as expected.

Table 3. Results of the computation

$n \times n$	2×2	4×4	8×8	16×16	32×32	64×64
$v_n \cdot 10^4$	55.278	68.282	82.665	97.296	111.989	126.695

So, the general conclusion is: joining bar elements and plane elasticity finite elements in points results in an incorrect statement of the problem.

What corrections should be made to the original mechanical model or the mathematical statement of the problem (which is the same thing) in order to describe the real structure behavior adequately? As it was stated before, to do that one should consider the design of a joint between a bar and a diaphragm in sufficient detail.

For example, let a steel I-beam be embedded partly in a brick wall as shown in Fig.6,*a*. Then it suffices to add a one-dimensional bar element that penetrates into the plane stressed wall by an appropriate part of its length, to the respective design model and into its discrete counterpart, as shown in Fig. 6,*b*.

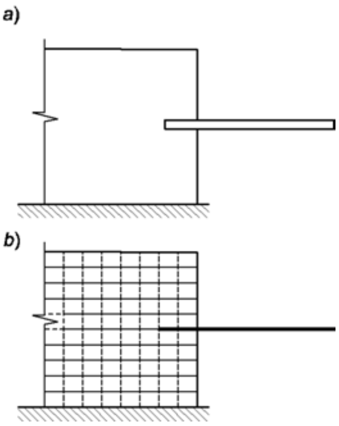

Figure 6: A cantilever clamped in a wall

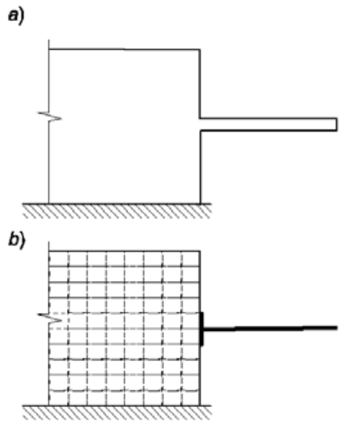

Figure 7: A monolithic joint

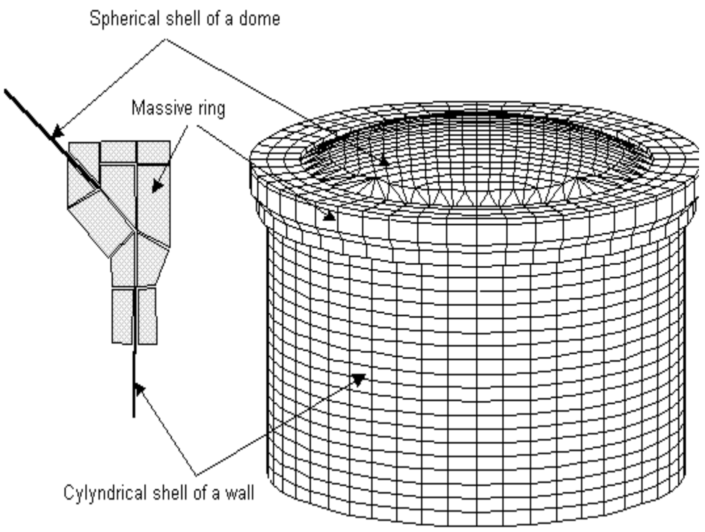

Figure 8: Coupling a reactor containment's shell and cover with a massive ring.

Another design can be suggested for a monolithic joint between a ferroconcrete wall panel and a spandrel beam of a building — see Fig. 7,*a*. Here one can take into account actual dimensions of the respective cross-section: the height of the beam, with a perfectly rigid body as high as the cross-section of the beam placed along the wall's border as shown in Fig. 7,*b*. This rigid body complies with the Bernoulli's plane section assumption for the beam itself. The assumption states that the beam's section must remain flat and must not change its size after its deformation.

Certainly the two ways of building design models given here by no means comprise the wide scope of possible situations. In every particular case an analyst should take into account peculiarities of his structure's design rather than some fancied and questionable lockstep patterns.

Also, it should be noted that the "joint area extension" technique as in Fig. 5 could be used to match other types of elements of different dimensionality. Fig. 8 shows an example wherein a cylindrical shell of a wall and a spherical shell of a cover of nuclear reactor containment are matched with a massive ring. It should be noted that in places of interpenetration of the structures some perturbation of rigidity properties arises due to a summation of the shell's and the massive body's rigidities in parts that occupy the same region of space. This fact should be taken into account when assigning the rigidity constants.

4. Bars + Massive Elements.

Problems where both bars and massive elements are involved to simulate structures include, for example, a joint analysis of a building's framework and massive foundations under columns. Issues of forming correct design models are similar to those described in the previous section, and we won't give any more details here. Just as in the case of coupling bars and plane stress elements, a formal nodal coupling is not applicable. The design model should be corrected so as to allow for particularities of the joint between the bar and the bulk body. Further details are omitted.

5. Plane Stress + Flexural Plates (Coupling Shell Elements).

The analysis of spatial plate structures, folded shells, and smooth shells approximated by piecewise-linear surfaces requires a coupling of plane stress elements and flat flexural members joint along their common boundary at an angle α (Fig. 9). Sometimes every finite element that participates in a discrete model can be a plane one and a flexural plate at the same time (such one is often called a plane shell element). This kind of elements has five degrees of freedom in each node.

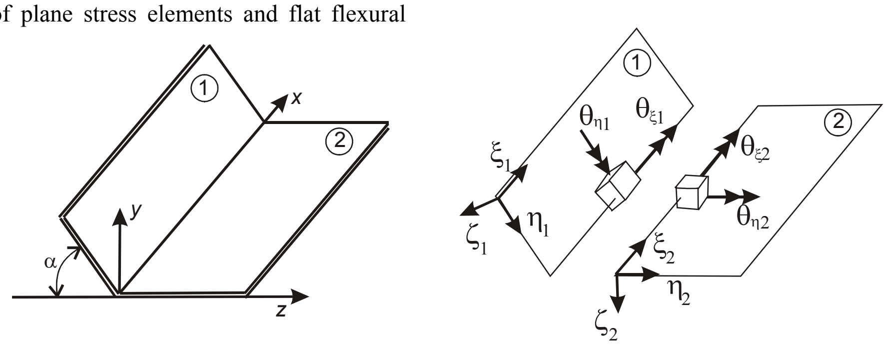

Figure 9: Coupling plane shell elements at an angle

Let us introduce a local Cartesian right-oriented coordinate system (ξ, η, ζ) referred to a particular plane element so that axes ξ, η belong to the median surface of the element and axis ζ is orthogonal to the former two axes.

Let u, v, w be linear displacements of points of the median surface of the shell with respect to the local coordinate system, θ_ξ, θ_η be slopes of the normals to the shell's median surface about respective axes ξ, η, and

$$\theta_\xi = \frac{\partial w}{\partial \eta}, \quad \theta_\eta = -\frac{\partial w}{\partial \xi}. \tag{9}$$

These 5 parameters — u, v, w, θ_ξ, θ_η (more exactly, values of these functions in nodal points of finite elements) — correspond to five degrees of freedom of each plane shell element node.

Let (X, Y, Z) be a global coordinate Cartesian system which we assume to be an orthogonal Cartesian right-oriented system. Introduce a matrix of direction cosines Λ that defines the mutual orientation of the global and local coordinate systems. We have:

$$\mathbf{\Lambda} = \begin{array}{cccc} \xi & \eta & \zeta & \\ \lambda_{x\xi} & \lambda_{x\eta} & \lambda_{x\zeta} & X \\ \lambda_{y\xi} & \lambda_{y\eta} & \lambda_{y\zeta} & Y \\ \lambda_{z\xi} & \lambda_{z\eta} & \lambda_{z\zeta} & Z \end{array}.$$

Now the components of the full nodal slope vector $\boldsymbol{\theta}$ of a particular finite element with respect to the global coordinate system (θ_x, θ_y, θ_z) can be expressed via its components referred to the local coordinate system (θ_ξ, θ_η, 0):

$$\theta_x = \lambda_{x\xi}\theta_\xi + \lambda_{x\eta}\theta_\eta + \lambda_{x\zeta}0,$$

$$\theta_y = \lambda_{y\xi}\theta_\xi + \lambda_{y\eta}\theta_\eta + \lambda_{y\zeta}0,$$

$$\theta_z = \lambda_{z\xi}\theta_\xi + \lambda_{z\eta}\theta_\eta + \lambda_{z\zeta}0.$$

If the α angle at which two adjacent shell planes (Fig. 8) cross is neither 0 nor π, then any node belonging to the intersection of the planes will acquire three linearly independent slopes, i.e. three rotational degrees of freedom. As can be seen in Fig. 9, two slope couples ($\theta_{\xi 1}$, $\theta_{\eta 1}$) and ($\theta_{\xi 2}$, $\theta_{\eta 2}$) are not coplanar (do not lie in the same plane) therefore form three linearly independent rotations in the three-dimensional space.

Thus, a node belonging to the boundary will have all six degrees of freedom. Now let us suppose a couple of planes intersect at the α angle equal to π. Then the node at the joint of the planes will lose one of three rotational degrees of freedom because all four slopes ($\theta_{\xi 1}$, $\theta_{\eta 1}$) and ($\theta_{\xi 2}$, $\theta_{\eta 2}$) are coplanar and do not form components of a rotation vector orthogonal to this plane.

It is here where the first issue of coupling plane shell elements arises. Some nodes acquire six degrees of freedom while the others only five. Some most advanced software developments can handle this situation by providing each node automatically only with degrees of freedom allowed by all finite elements adjoining the node. In other cases the user has to control this process manually by imposing external constraints on nodal slopes not resisted by adjoining elements.

Let us now return to our problem of coupling plane shell elements, supposing that somehow we have contrived to solve the problem of matching the nodal degrees of freedom by their number. For the sake of simplicity, assume that two intersecting planes are orthogonal, i.e. the α angle is equal to $\pi/2$ in Fig. 9. Then a fragment of the intersection line of the two planes located between two nearest nodes of the line will belong to two finite element simultaneously, and thus it will acquire inter-nodal displacements distributed by a certain law in one element (say, a linear law when the boundary moves in the element's plane) and by a different one in the other element (say, a polynomial law if the boundary moves out of the element's plane). It's clear enough this situation will result in discontinuity of the displacements. In other words, flexural plate finite elements and plane stress plate finite elements that intersect (at an angle other than π) will be inconsistent. And the very first question immediately and naturally asked by the user goes like this: how dangerous is this inconsistency from the standpoint of violation of the finite element solution convergence?

A qualitative description of the effect of the said inconsistency on the finite element solution convergence can be obtained from the following simple consideration.

Let us introduce a global coordinate system (X, Y, Z) in our problem where two adjoining planes intersect, as shown in Fig. 9. Let $u_1(x, y)$, $v_1(x, y)$, $w_1(x, y)$ be the displacements of points of the plane 1 in the directions of axes X, Y, Z, respectively, and $u_2(x, y)$, $v_2(x, y)$, $w_2(x, y)$ the displacements of points of the plane 2 in the same directions. Then, the coupling conditions at the common boundary of the two planes can be stated as

$$u_1(x,0) = u_2(0,x),\ \ v_1(x,0) = v_2(0,x),$$

$$w_1(x,0) = w_2(0,x),\ \ \theta_{1x}(x,0) = \theta_{2x}(0,x),. \tag{10}$$

where $\theta_{1x}(x,0) = \frac{\partial w_1}{\partial y}(x,0),...\theta_{2x}(0,x) = -\frac{\partial v_2}{\partial z}(0,x)$.

And that's all folks! Any additional coupling conditions not following directly from relations (10) lead to a mathematically incorrect problem statement.

It is clear the displacement discontinuities that bother us so much are due to a violation of the second and third coupling conditions (10). It suffices to choose one of them, say, the second one which holds in nodal points belonging to the two planes intersection line and does not between the nodes. We mark coordinates and displacement components referring to two adjacent nodes that belong to the said line of intersection of the two planes, with indices i and i+1. Then for all $x \in [x_i , x_{i+1}]$, assuming

$$l = x_{i+1} - x_i, \qquad \xi = (x - x_i) / l ,$$

we have

$$v_1(x, 0) = v_i\, L_1(\xi) + v_{i+1}\, L_2(\xi) ,$$

$$v_2(0,x) = v_i\, H_1(\xi) + \theta_{i,z}\, l\, H_2(\xi) + v_{i+1}\, H_3(\xi) + \theta_{i+1,z}\, l H_4(\xi) , \tag{11}$$

where an additional notation is included:

$L_1(\xi) = 1- \xi$, $L_2(\xi) = \xi$ are linear functions,
$H_1(\xi) = 2\xi^3 - 3\xi^2 + 1$, $H_2(\xi) = \xi^3 - 2\xi^2 + \xi$,
$H_3(\xi) = -2\xi^3 + 3\xi^2$, $H_4(\xi) = \xi^3 - \xi^2$ are Hermitian polynomials,

$\theta_{i,z}$, $\theta_{i+1,z}$ are rotations/slopes of respective nodes i and i+1 about the Z axis,

v_i , v_{i+1} are linear displacements of nodes i and i+1 in the Y axis direction.

Note that the expression of $v_2(0,x)$ in (11) holds for rectangular Adini – Clough elements [2].

As the mesh becomes denser, the distance l between adjacent nodes tends to zero, and the nodal slopes $\theta_{i,z}$ and $\theta_{i+1,z}$ can be replaced by difference relations becoming more accurate with less l. Taking into account the rule of signs for the slopes, we can write

$$\theta_{i,z} = \theta_{i+1,z} = (v_{i+1} - v_i)/l . \tag{12}$$

Now replacing slopes in (11) by the difference relations from (12), we have $v_1(x,0) = v_2(0,x)$, that is, discontinuities in the displacements are disappearing as the spacing of the mesh approaches zero.

Software developers have to attempt solving two problems at the same time:

(1) match the number of degrees of freedom of nodes both belonging and not belonging to the same plane together with all adjoining plane shell elements;

(2) eliminate the inconsistency of displacements of elements joined at an angle other than zero or π.

With this purpose, some programs suggest plane shell finite elements with a sixth degree of freedom, and this one is treated geometrically as a rotation of the node in the plane of a finite element. The inconsistency of the displacements of these elements cannot be fully eliminated, but the convergence of the finite element solution is ensured by the conditions of piecewise testing. Though, the software actually requires six coupling conditions to be satisfied along the intersection line of the two planes, instead of the four from (10), and two more coupling conditions violate the mathematical correctness of the problem. To illustrate this, let's return to our example of an orthogonal intersection of two planes.

Introducing a rotation component ω_{1z} for the plane 1 (say, as an average rotation in the vicinity of the point in question, i.e. a node), and demanding that ω_{1z} be equal to the slope θ_{2z} of the normal to the median surface in the plane 2, along the common boundary of the two planes, we have:

$$\frac{1}{2}\left[\frac{\partial v_1}{\partial x}(x,0) - \frac{\partial u_1}{\partial y}(x,0)\right] = \frac{\partial v_2}{\partial x}(0,x) \tag{13}$$

wherefrom, taking into account (10), we come up with a condition of no shear in the plane 1 where it joins the plane 2, that is

$$\gamma_{1xy}(x,0) = \frac{\partial v_1}{\partial x}(x,0) + \frac{\partial u_1}{\partial y}(x,0) = 0 \tag{14}$$

In the same way we obtain the second additional condition along the junction line of the two planes:

$$\gamma_{2xy}(0,x) = \frac{\partial u_2}{\partial z}(0,x) + \frac{\partial w_2}{\partial x}(0,x) = 0$$

It should be clear that relations (14) and (15) are conditions of a mutual slip of the two planes along the X coordinate axis, that is, they contradict the first coupling condition of (10) interpreted mechanically as conditions of "adhesion" of the two planes.

So we repeat once more: there is nothing bad about introducing additional sixth degree of freedom, but the user should understand its meaning clearly and not identify it formally with the slope of the respective node. It is of particular importance for the formulation of boundary conditions and conditions of coupling of different plates of a folded shell intersecting at an angle.

References

[1] Timoshenko, S.P. and Voinovsky-Kriger, C. Theory of plates and shells, McGraw-Hill, New York, Toronto, London, 1959.

[2] Adini, A. and Clough, R.W. Analysis of Plate Bending by the Finite Element Method: Report submitted to the National Science Foundation, Grant G7337, 1960.