**MODERN BUILDING MATERIALS, STRUCTURES AND TECHNIQUES**

May 19-21, 2004
The 8th International Conference

**Faculty of civil Engineering,**
**Vilnius Gediminas Technical University**
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# APPLICATION OF ITERATIVE SOLVERS IN FINITE ELEMENT ANALYSIS OF STRUCTURAL MECHANICS. LINEAR STATICS AND NATURAL VIBRATIONS

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**Abstract.** The family of iterative methods for static and natural vibration analysis, based on preconditioned conjugate gradient (PCG) method with aggregation multilevel preconditioning, is considered. Both: the element-by-element procedure for assembling of stiffness matrix and sparse direct solver for it factoring and fast forward – backward substitutions ensure the high stability of methods against ill-conditioning. The generalized preconditioned conjugate gradient method with shifts in aggregation multilevel preconditioning is developed to overcome the lock of convergence, which is met when conventional PCG methods are applied for eigenvalue analysis.

**Keywords:** Preconditioned conjugate gradient method; Aggregation approach; Element-by-element technique; Shift in preconditioning

## 1. Introduction

This paper is devoted to application of iterative methods to analysis of large-scale finite element problems of structural mechanics. The multi-storey civil of industrial buildings, thin-walled structures and other objects give rise the poorly - conditioned finite element (FE) problems. Usually direct methods, based on $\mathbf{LU}$, $\mathbf{LDL}^T$, $\mathbf{LL}^T$ factoring, are used for such problems. The size of FE problem is enough big, the direct methods are more and more consuming time.

The results, presented on fig. 1, illustrate the following: the dimension of FE model is bigger the iterative solution is more preferable comparing with direct one. The conventional skyline solver and sparse direct multifrontal solver MFM [6] are compared with aggregation multilevel iterative one AMIS [2, 3, 5]. So, each task possesses such a dimension when iterative solver requires less computational efforts than a direct one.

However, the structural mechanics problems usually are poorly conditioned and conventional iterative solvers are inefficient for their analysis.

The preconditioning is a powerful approach for solution of ill-conditioned problems. The preconditioned conjugate gradient method with aggregation multilevel preconditioning is presented here.

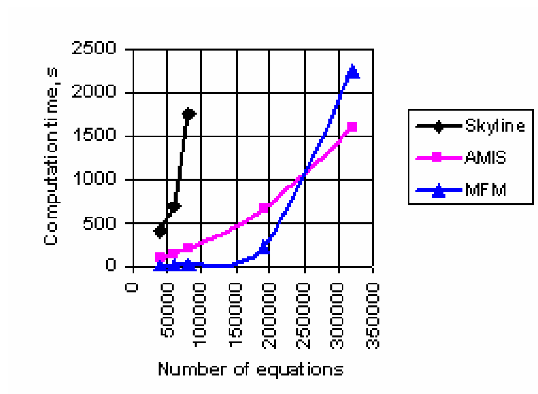


**Fig. 1** Computation time via number of equations for direct (skyline, MFM) and iterative (AMIS) solvers

The idea of aggregation multilevel iterative method is presented in [1]. In [2, 3, 5] and in this paper we present the aggregation approach on the base of element-by-element technique which is applied to assembling of coarse level matrix (stiffness matrix of aggregation model), and sparse direct technique, applying to factorization of this matrix. It is essential peculiarities of presented here aggregation multilevel method, because the large number of equations of coarse level model is kept and a good prediction of a slow converged low solution modes is ensured.

The multi-storey buildings, the objective – oriented mesh generators, the strong requirements for accuracy of numerical results and other reasons lead to arising of large-scale finite element models.

The family of high-performance iterative methods for static and natural vibration analysis of large-scale finite element models is presented. These methods are based on preconditioned conjugate gradient method with aggregation multilevel preconditioning, element-by-element technique of coarse level matrix preparation and sparse direct technique which is applied to factoring of coarse level matrix and to the forward-backward substitutions during PCG iterations.

## 2. An aggregation multilevel preconditioning

The basic idea of the multilevel preconditioning for the PCG method is presented below.

First, a coarse level model is created. Then, the restriction-prolongation operators $\mathbf{Q}^T, \mathbf{Q}$ are formulated to establish an interaction between the coarse and fine level models. The procedure presented below is applied instead of an explicit solution

$$\mathbf{B}\mathbf{z}_k = \mathbf{r}_k \quad (1)$$

where $\mathbf{r}_k$ is a residual vector, $k$ (an iteration number) will be omitted.

- Restriction of the $\mathbf{r}$ vector to the coarse level: $\mathbf{r}_f \mapsto \mathbf{r}_c$. This procedure consists of transforming the fine level model into the coarse level: $\mathbf{r}_c = \mathbf{Q}^T \mathbf{r}_f$ and $\mathbf{Q}^T$ is the restriction operator. The upper subscript $^T$ denotes a transposition, lower subscripts $f, c$ refer to respective fine and coarse level models.
- Resolution of $\mathbf{K}_c \mathbf{z}_c = \mathbf{r}_c$, where $\mathbf{K}_c = \mathbf{Q}^T \mathbf{K} \mathbf{Q}$ ($\mathbf{K}_c$ is already decomposed and the size of the coarsest level problem allows the implementation of the direct methods).
- Prolongation $\mathbf{z}_c \mapsto \mathbf{z}_f^*$ from the coarse level to the fine level. This operation consists of a transformation from the coarse level model into the fine level: $\mathbf{z}_f^* = \mathbf{Q}\mathbf{z}_c$ and $\mathbf{Q}$ is the prolongation operator
- Smoothing of the vector $\mathbf{z}_f^* \mapsto \mathbf{z}_f$ after the prolongation. Rapidly fluctuating residuals appear during the prolongation. An internal iteration procedure is applied to damp the residuals.

The implementation of the aggregation approach presented here is based on element-by-element technique used to prepare the coarse level matrix $\mathbf{K}_c$ promptly. A more efficient PCG algorithm and element-by-element aggregation scheme [2], [3], [5] allows us to improve the robustness of the method and incorporate it in the Robot Millennium commercial software (www.robobat.com). Now this method is incorporated in the SCAD software (www.scadgroup.com).

The aggregation approach consists of an introduction of additional connections (rigid links) to decrease the number of degrees of freedom of a given design model. The coarse level model is derived as shown below (Fig.2). Thus, the original finite-element model (fine level) is transformed into a mechanical system (coarse level), which consists of non-overlapped local rigid aggregates coupled by elastic connections. The rigid aggregates are rigid bodies due to the imposed rigid links. All nodes of the finite-element model should be included in the rigid aggregates. It is possible to treat a single node as a limit case of a minimal rigid aggregate. It is not admissible for any node to be included into more than one aggregate.

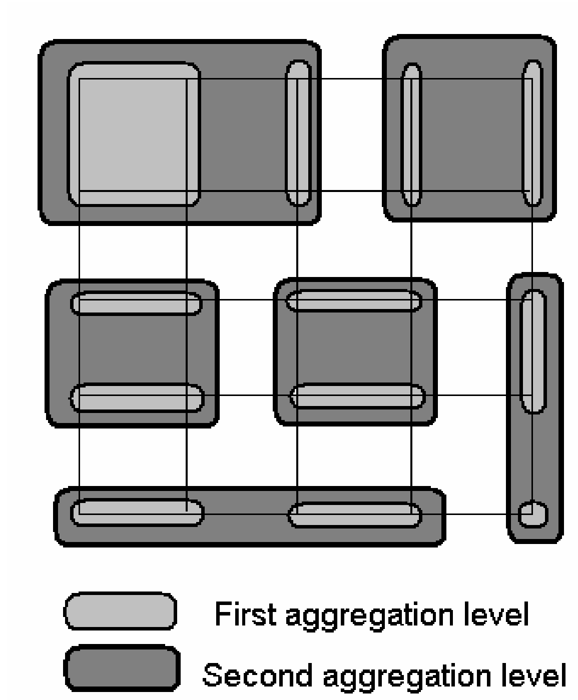


**Fig.2** First and second aggregation levels for a rectangular plate with the finite element mesh 4×4

The first aggregation step is performed in an element-by-element loop. We take the first finite element and couple all nodes belonging to it. Aggregated nodes are marked. Then we take the second element and couple the remaining (unmarked) nodes. And so on.

The second aggregation level (and all the following) is performed in the same way. The aggregates from the previous level are considered to be generalized nodes. Each aggregate from the previous level which is coupled into a new aggregate of the current level, is marked to avoid a total coupling of the entire structure.

This aggregation procedure is being applied until the size of the coarsest level model becomes small enough for a direct solution. This approach keeps the topological similarity of each aggregation level to the original model (fine level).

The details are presented in [2], [3], [5].

## 3. The static analysis

The several large-scale problems are considered. The skyline solver, incomplete Cholesky factorization solver ICCG, sparse direct multifrontal solver MFM [6] and aggregation multilevel iterative solver AMIS are compared. The skyline and ICCG solvers are widely used in commercial finite element programs, and therefore we account such methods as traditional ones.

### 3.1. Example 1 (multi-storey building — 544 410 equations)

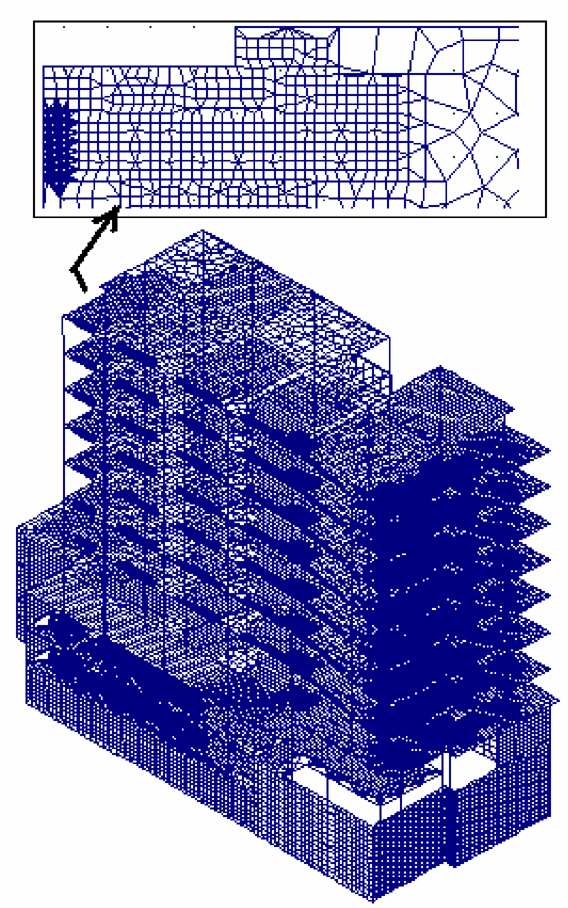

**Fig. 3** Multi-storey building — 544 410 equations

The non-uniform mesh on slabs and the strong difference of stiffness between several parts of structure cause the ill-conditioning. The computation time for several methods and number of iterations for iterative solvers is shown in Tab. 1.

The precision of iterative methods is taken as

$$\|\mathbf{b}-\mathbf{Kx}\|_2 / \|\mathbf{b}\|_2 < tol \wedge \|\mathbf{b}-\mathbf{Kx}\|_\infty / \|\mathbf{b}\|_\infty < tol \qquad (2)$$

where $tol = 10^{-3}$. Such a precision is a quite enough for engineering purposes. The Pentium III computer (CPU Intel 1000 MHz, 512 MB RAM) has been used for all methods.

**Table 1**. Performance of presented methods.

| Method | Computation time | Number of iterations |
|---|---|---|
| Skyline | 23 h 30 m | — |
| ICCG | 2 h 59 m | 6 815 |
| MFM | 36 m | — |
| AMIS | 17 m | 70 |

The presented here AMIS method allows to reduce the computation time almost in 2 times comparing with even sparse direct multifrontal solver.

The comparison of convergence for AMIS and ICCG methods is presented on Fig. 4.

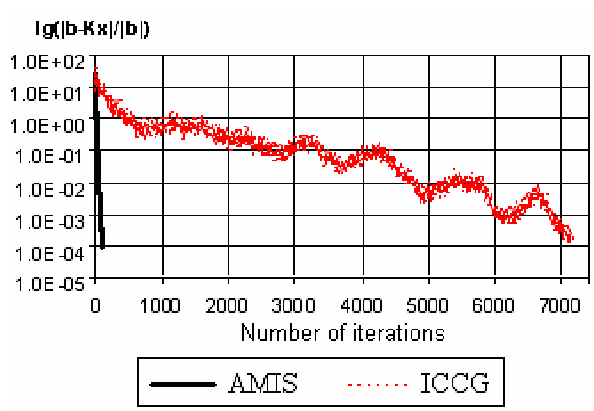


**Fig. 4** Convergence of AMIS and ICCG methods

The large number of iterations for ICCG method evidences that this problem is poorly conditioned.

### 3.2. Example 2 (multi-storey building — 1 171 104 equations)

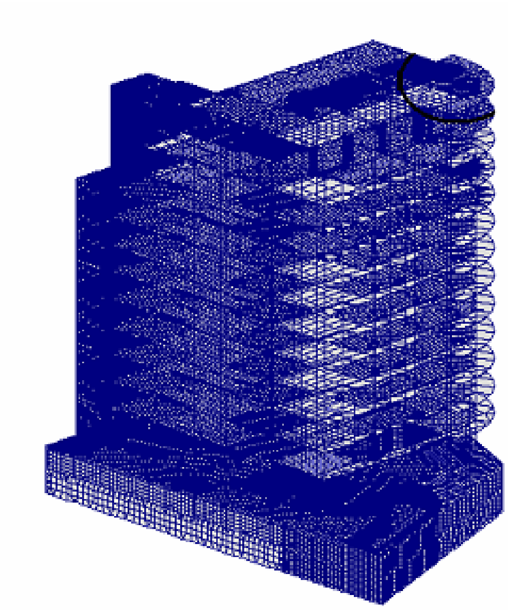

**Fig. 5** Multi – storey building — 1 171 104 equations

Three load cases (right-hand sides) is considered. The Pentium-IV computer (CPU Intel 2.8 GHz, RAM 1024 MB) has been used.

The computation time for several methods and number of iterations for iterative solvers is shown in Tab. 2. The number of iterations for each load case is separated by slash.

**Table 2**. Performance of presented methods.

| Method | Computation time | Number of iterations |
|---|---|---|
| Skyline | ~ 48 h | — |
| ICCG | 1 h 24 m | 2507 / 1926 / 2506 |
| MFM | 2 h 19 m | — |
| AMIS | 20 m | 45 / 45 / 42 |

The convergence of iterations for AMIS and ICCG methods for first load case is shown on Fig. 6.

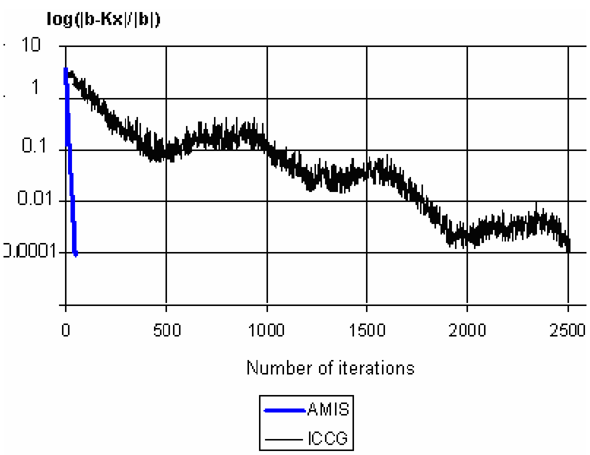


**Fig. 6** Convergence of AMIS and ICCG methods

The skyline solver requires 21 606 MB memory on hard disk for allocation of factorized stiffness matrix. It is impossible to store such a large matrix to disk for our computer. Therefore we can only estimate the computation time for skyline solver.

The multifrontal sparse direct solver MFM requires 3 246 MB HDD memory due to drastic decreasing of nonzero entries in factorized stiffness matrix.

The AMIS solver occurs most preferable and allows us to reduce essentially the analysis time.

**4. Natural vibrations analysis.**

The generalized conjugate gradient method with shifted aggregation multilevel preconditioning [3, 5] is applied to analysis of large-scale natural vibrations finite element problems. This method combines the advantages of PCG approach for generalized eigenvalue solution and shift technique which allows us to accelerate the convergence and avoid the lock of it when close eigenvalues are met.

We compare the performance of MPCG_AMIS (modified preconditioned conjugate gradient method with aggregation multilevel preconditioning) with ones of conventional PCG_ICCG method (preconditioned conjugate gradient method with incomplete Cholesky factorization preconditioning without shifts) and block Lanczos method [4], based on sparse direct multifrontal solver [6].

**4.1 Example 1 (multi-storey building — 544 410 equations)**

The computation model is shown on Fig. 3. The five eigenpairs are extracted. The precision of computations for iterative methods is defined as following:

$$err = \frac{\|\mathbf{K}\mathbf{x}_i - \lambda_i \mathbf{M}\mathbf{x}_i\|_2}{\|\lambda_i \mathbf{M}\mathbf{x}_i\|_2} < tol \qquad (3)$$

where $tol = 10^{-3}$, $\mathbf{K}, \mathbf{M}$ — stiffness and mass matrices, $\{\lambda_i, \mathbf{x}_i\}$ — eigenpair for i-th mode. The Pentium III computer (CPU 1266 MHz, 512 MB RAM) has been used.

The computation time for several methods and number of iterations for iterative solvers is shown in Tab. 3.

**Table 3**. Performance of presented methods.

| Method | Computation time | Number of iterations | Nonzero entries, MB |
|---|---|---|---|
| Lanczos (Skyline) | >> 23 h | — | 29 166 |
| PCG_ICCG | — | > 13 000 | — |
| Block Lanczos (MFM) | 37 m 25 s | | 763 |
| MPCG_AMIS | 33 m 52 s | 242 | — |

The Lanczos method, based on skyline solver, is fault due to large size of nonzero entries for factorized stiffness matrix. The PCG_ICCG method exhibits the lock of convergence for second mode. The block Lanczos method [4], based on sparse direct multifrontal solver, and presented MPCG_AMIS method allows us to obtain the efficient solution of this problem.

**4.2. Example 2 (multi-storey building — 1 171 104 equations)**

The computation model is shown on Fig. 5. The five eigenpairs is extracted. The precision of computations for iterative methods (see (3)) is taken as $tol = 10^{-3}$. The computation time for several methods and number of iterations for iterative solvers is shown in Tab. 4. The Pentium IV computer (CPU 2.8 GHz, 1024 MB RAM) has been used.

**Table 4**. Performance of presented methods.

| Method | Computation time | Number of iterations | Nonzero entries, MB |
|---|---|---|---|
| PCG_ICCG | Lock of convergence on first mode | | — |
| Block Lanczos (MFM) | 4 h 01 m | | 3 246 |
| MPCG_AMIS | 49 m | 422 | — |

The conventional PCG_ICCG method is fault due to lock of convergence on the first mode. The MPCG_AMIS method allows us to reduce the analysis time comparing with block Lanczos method [4] in 4 times. The convergence of iterations for each mode is presented on Fig.7.

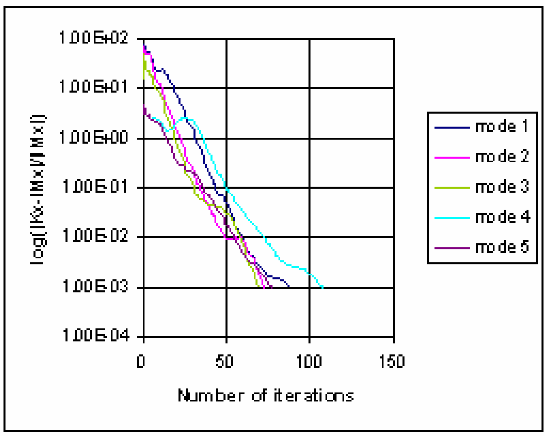


**Fig. 7** Convergence of modes for MPCG_AMIS method

## 6. Summary

The family of iterative methods, presented here, is based on preconditioned conjugate gradient approach with aggregation multilevel preconditioning, exhibits a high performance and robust convergence. The element-by-element technique of coarse stiffness matrix preparation and sparse direct approach for its factoring allows us to keep a large size of aggregation model (till 200 000 equations for direct solution on PC computers). It ensures the high stability against ill-conditioning and robust convergence.

The generalized preconditioned conjugate gradient method with shifts in aggregation multilevel preconditioning allows us to overcome the lock of convergence, which is met for conventional PCG methods for eigenproblem solution.

These methods are possible to be applied to analysis of large-scale poorly-conditioned finite element models of structural mechanics, particularly, to FEM models of multi-storey buildings.

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