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On an Error of a Mysterious Nature that Happens in Software when Analyzing Mechanical Systems for Buckling

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Summary Unexpectedly erroneous solutions have been found practically in all known programs when analyzing constructions for stability. It has been discovered that these mistakes result from the incorrect statement of the stability problem for systems containing perfectly rigid bodies. General algorithm leading to correct solutions is introduced.

Introduction

As is well known neither testing nor long experience of engineering software can give a 100% guaranty for its being «bugless». Much more seldom (and more treacherous and dangerous!) are the mistakes caused by overlooking in the basis of the algorithm. One of these unpleasant mistakes which is able to lead to catastrophic sequences penetrated into commercial software. The aim of this paper: 1) – to warn the users 2) – to demand from the developers to eliminate these bugs using the algorithm suggested.

Examples of errors

Let us begin with a simplest problem of buckling analysis of a mechanical system shown schematically in Fig. 1. The system includes a cantilever bar of length l with its flexural rigidity EI and a perfectly rigid insert of length h at its end.

Let us now try to solve two problems using a software tool. Let's take actual numerical values and assume $EI = 1$, $l = 1$, $h = 1$ for determinacy. Strange as it seems, but all computational programs checked by us have yielded the same result: a correct solution of the problem in Fig. 1 - *a* and an incorrect one of the problem in Fig. 1 - *b*: $P_{crA} = P_{crB} = 2.467$. The correct solution must give $P_{crB} = 0.741$. So, in the case considered the critical load for the problem in Fig. 1 - *b* found by the software is upper than the proper value more than three times!

A dramatically more erroneous result is yielded by most commercial CAD programs for the problem shown in Fig. 2. Alas! But all programs we tested, except for the **SCAD** software [1], state that the system does not buckle at all. Without considering the whole process of solution of this simple problem, let's point out the correct value of the critical load $P_{cr}=3EI/(lh)$.

After some thinking one can easily conclude that the developers of most commercial programs have been fooled by the same trick, at the same time trying to have their users swallow that dangerous bait.

The matter is that a perfectly rigid body cannot be treated as a linear constraint in buckling problems. Indeed, in the buckling analysis the equilibrium equation systems must be formulated as linearized (rather than linear!). In other words, the equilibrium equations should be derived for a deformed state of the system, thus corresponding to the weakest form of geometrical nonlinearity.

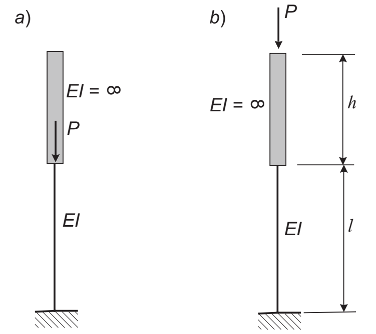

Figure 1: A buckling analysis of a cantilever bar

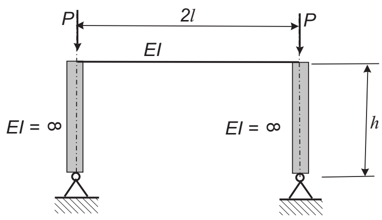

Figure 2: A test problem for the buckling analysis

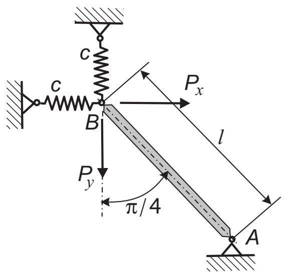

Figure 3: An example of determining the critical load

Correct general algorithm

So, let a mechanical system include a perfectly rigid body. Let's assume that a transition from the given mechanical system to its discrete design model has been done already. If now we extract the rigid body from the said model, then the action of the removed part of the model upon this body can be replaced by appropriate reactive forces and moments. In order to think generally, let's assume that besides the reactive forces the rigid body is subjected to some given (active) concentrated forces and moments. Let's denote the total number of points (or nodes, to one's liking) of the rigid body as m, vector forces $\boldsymbol{P}_i$ and vector moments $\boldsymbol{M}_i$ ($i = 1,\ldots,m$) being applied to the points.

Now let's choose an arbitrary point O in this body and call it a *pole*. Particularly, though not obligatorily, the pole may coincide with one of m already selected nodes. Let $\boldsymbol{r}_i$ be a radius vector that goes from the pole to i-th node. Recalling that the stability of some **balanced** state of the system is under consideration, we can conclude that the extracted rigid body under all applied (both active and reactive) forces $\boldsymbol{P}_i$ and moments $\boldsymbol{M}_i$ ($i = 1,\ldots,m$) must be in the state of equilibrium also. The latter fact means that the general force vector and the moment vector referred to the pole must be equal to zero:

$$\sum_{i=1}^{m} \boldsymbol{P}_i = \boldsymbol{0}, \quad \sum_{i=1}^{m} \left(\boldsymbol{M}_i + \boldsymbol{r}_i \times \boldsymbol{P}_i \right) = \boldsymbol{0}. \tag{1}$$

In the course of buckling, the displacements of a rigid body can be unambiguously determined by the pole displacement vector $\mathbf{u}_0$ and the body rotation vector $\boldsymbol{\theta}$, both beginning at the pole O. The full displacement $\mathbf{u}_i$ of the node i, taking into account the $\mathbf{u}_0$ node displacement, will look as follows

$$\mathbf{u}_i = \mathbf{u}_0 + \sin\theta\, \mathbf{e}\times \boldsymbol{r}_i + (\cos\theta - 1)\,[\, \boldsymbol{r}_i - \mathbf{e}\,(\mathbf{e}\cdot\boldsymbol{r}_i)]\,, \tag{2}$$

where ort $\mathbf{e}$ is directed along the axis of the body's revolution. Equation (2) is followed from the formula of Rodrigues [2]. Expanding the trigonometric functions into Taylor series and keeping only second-order and lower terms we derive a simplified constraint equation in the following form:

$$\mathbf{u}_i = \mathbf{u}_0 + \theta\, \mathbf{e}\times \boldsymbol{r}_i - \tfrac{1}{2}\,\theta^2\,[\, \boldsymbol{r}_i - \mathbf{e}\,(\mathbf{e}\cdot\boldsymbol{r}_i)]\,, \tag{3}$$

Now let's calculate the work A of all forces applied to the perfectly rigid body. We have

$$\mathsf{A} = \sum_{i=1}^{m} (\boldsymbol{P}_i \bullet \mathbf{u}_i + \boldsymbol{M}_i \bullet \boldsymbol{\theta}) \,. \tag{4}$$

By substituting the expression for $\mathbf{u}_i$ from (3) to (4) we find that the work A consists of two quantities $\mathsf{A} = \mathsf{A}_1 + \mathsf{A}_2$, where A_1 depends linearly on the components of vectors $\mathbf{u}_0$ and $\boldsymbol{\theta}$, while A_2 is a homogeneous quadratic form with respect to the components of the rotation vector $\boldsymbol{\theta}$.

$$\mathsf{A}_1 = \sum_{i=1}^{m} [\boldsymbol{P}_i \bullet \mathbf{u}_0 + (\boldsymbol{M}_i + \boldsymbol{r}_i \times \boldsymbol{P}_i) \bullet \boldsymbol{\theta})] \,, \quad \mathsf{A}_2 = \sum_{i=1}^{m} [-\frac{1}{2}\theta^2 (\boldsymbol{P}_i \bullet \boldsymbol{r}_i) + \frac{1}{2} (\boldsymbol{P}_i \bullet \boldsymbol{\theta})(\boldsymbol{r}_i \bullet \boldsymbol{\theta})] \,. \tag{5}$$

First of all let us notice that the conditions of A_1 stationarity lead expectedly to equilibrium equations (1) for perfectly rigid bodies.

Further, the components of the geometrical stiffness matrix $\mathbf{K}_{\mathrm{G}}$ produced by the rigid body will be determined by double differentiation of the expression for work A. In other words,

$$\mathbf{K}_{\mathrm{G}} = \left\| \begin{matrix} g_{\theta_x\theta_x} & g_{\theta_x\theta_y} & g_{\theta_x\theta_z} \\ g_{\theta_y\theta_x} & g_{\theta_y\theta_y} & g_{\theta_y\theta_z} \\ g_{\theta_z\theta_x} & g_{\theta_z\theta_y} & g_{\theta_z\theta_z} \end{matrix} \right\| = -\left\| \begin{matrix} \dfrac{\partial^2 \mathsf{A}}{\partial\theta_x \partial\theta_x} & \dfrac{\partial^2 \mathsf{A}}{\partial\theta_x \partial\theta_y} & \dfrac{\partial^2 \mathsf{A}}{\partial\theta_x \partial\theta_z} \\ \dfrac{\partial^2 \mathsf{A}}{\partial\theta_y \partial\theta_x} & \dfrac{\partial^2 \mathsf{A}}{\partial\theta_y \partial\theta_y} & \dfrac{\partial^2 \mathsf{A}}{\partial\theta_y \partial\theta_z} \\ \dfrac{\partial^2 \mathsf{A}}{\partial\theta_z \partial\theta_x} & \dfrac{\partial^2 \mathsf{A}}{\partial\theta_z \partial\theta_y} & \dfrac{\partial^2 \mathsf{A}}{\partial\theta_z \partial\theta_z} \end{matrix} \right\| . \tag{6}$$

It is this geometrical stiffness matrix which is not taken into consideration by programs that yielded the erroneous result in the examples given above.

Expanding the expression (5) of A_2 into its coordinate form, we can write expressions for the components of our matrix of interest $\mathbf{K}_{\mathrm{G}}$ explicitly. So,

$$g_{\theta_x\theta_x} = \sum_{i=1}^{m} (P_{iy} r_{iy} + P_{iz} r_{iz}), \quad g_{\theta_y\theta_x} = -\sum_{i=1}^{m} (P_{ix} r_{iy} + P_{iy} r_{ix}), \quad g_{\theta_z\theta_x} = -\sum_{i=1}^{m} (P_{ix} r_{iz} + P_{iz} r_{ix}). \tag{7}$$

The components of the second and third columns of matrix $\mathbf{K}_{\mathrm{G}}$ are not given here as they can be derived from (7) by a circular permutation of its indexes.

In a particular case of a plane mechanical system, the $\boldsymbol{\theta}$ rotation vector is orthogonal to the plane of the system of interest (say, plane X,Y) and thus has only one component θ_z. The order of the geometrical stiffness matrix of the rigid body becomes one, and its only component becomes especially simple:

$$g_{\theta_z\theta_z} = \sum_{i=1}^{m} (P_{ix} r_{ix} + P_{iy} r_{iy}) = \sum_{i=1}^{m} \boldsymbol{P}_i \bullet \boldsymbol{r}_i \,. \tag{8}$$

Example

Let's demonstrate the technique of using the geometrical stiffness matrix of a perfectly rigid body by the example of an elementary problem shown in Fig. 3.

In this problem we assume that both springs satisfy a linear dependence between the stress in the spring and the respective displacements of the B point. First of all, let us find the distribution of internal forces in the system assuming it to be linearly deformable. It is easy to see that this system has only one degree of freedom due to the presence of a perfectly rigid bar. The said degree of freedom will be the angle of rotation θ of the rigid bar around point A, the fixation point of this bar. Further, in the case of small rotations (the geometrically linear formulation of the problem is being considered) the horizontal u and vertical v displacements of point B will be equal, respectively, to

$$u = \frac{\sqrt{2}}{2} l\theta\,, \quad v = -\frac{\sqrt{2}}{2} l\theta\,. \tag{9}$$

Assuming each spring's stiffness to be c, let's write out an expression for the full potential energy of the system

$$\mathrm{L} = \mathrm{L}(\theta) = \frac{cl^2\theta^2}{2} - \frac{l\sqrt{2}}{2}(P_x - P_y)\theta\,. \tag{10}$$

The condition of stationarity of L as a function of θ will yield the following equilibrium equation $d\mathrm{L}/d\theta = 0$ wherefrom

$$\theta = \frac{\sqrt{2}}{2cl}(P_x - P_y)\,. \tag{11}$$

The stresses in the horizontal spring N_x and vertical spring N_y will be respectively equal to $N_x = cu = ½\,(P_x - P_y)$, $N_y = cv = ½\,(P_y - P_x)$. It is easy to understand that these forces taken together with P_x and P_y can be reduced to one force $N = (P_x + P_y)/\sqrt{2}$ working on the extracted bar in the direction of the bar's axis and the positive value of the sum of external forces $(P_x + P_y)$ makes the longitudinal force N compressive. Now, following formula (9.42), let us find the component of the geometrical stiffness matrix produced by the rigid bar. We have

$$g_{\theta\theta} = -\frac{P_x + P_y}{\sqrt{2}}\, l\,. \tag{12}$$

The conventional stiffness matrix **K** of the system consists here of only one element $\mathbf{K} = |[k]|$, and

$$k = \frac{\partial^2 \mathrm{L}}{\partial\theta\partial\theta} = cl^2\,. \tag{13}$$

The critical state of the system is determined by the condition $\det(\mathbf{K} + \mathbf{K}_G) = 0$, which gives us in this case

$$(P_x + P_y)_{cr} = cl\sqrt{2}\,. \tag{14}$$

This completes the analysis of stability of the discussed system in the linearized formulation.

Concluding remarks

To conclude, we'd like to notice that there is certainly no mystery at all in the way one poses and solves buckling problems for systems that contain perfectly rigid bodies. We have used this word in the title only to emphasize a psychological aspect rather than physical one. What seems to us a mystery is the reason for which the developers of a number of software systems, who are surely beyond any suspicion of incompetence, have fell victims to a popular fallacy by forgetting the peculiarity of perfectly rigid bodies discussed here in the aspect of buckling. What a shame that the result of this fallacy should be a disoriented user of computational software. Hence our appeal to the community of developers that they should try to eliminate this fault in their commercial software as soon as possible, in order not to endanger the army of users by having them obtain incorrect results.

References

[1] http://www.scadgroup.com

[2] A. Lurie. *Analytical mechanics* (in Russian). Nauka, Moscow, (1961).