The book is devoted to the history of structural mechanics and structural theory presented in the form of essays on development of their trends; naturally, the list of these trends may be extended. Therewith, certain history of origin and formation of the corresponding concepts, principles, ideas and problems and ways of their solution is inherent in each trend. The presentation of content by sections is accompanied by facts from the life and activities of outstanding scientists as well as by cognitive illustrations. It is according to James Clerk Maxwell: “In Science, it is when we take some interest in the great discoverers and their lives that it becomes endurable, and only when we begin to trace the development of ideas that it becomes fascinating”. The book may be used as a manual for students of higher educational institutions realizing master’s programs, studying special course, etc. It is generally oriented to the students and readers, who have studied necessary courses of structural mechanics and related engineering disciplines as well as for lecturers and scientific-and-technical workers.

Viktor Bazhenov
Anatolii Perelmuter
Yurii Vorona

Structural mechanics and theory of structures. History essays
Viktor Bazhenov
Anatolii Perelmuter
Yurii Vorona

Structural mechanics and theory of structures. History essays
Imprint
Any brand names and product names mentioned in this book are subject to trademark, brand or patent protection and are trademarks or registered trademarks of their respective holders. The use of brand names, product names, common names, trade names, product descriptions etc. even without a particular marking in this work is in no way to be construed to mean that such names may be regarded as unrestricted in respect of trademark and brand protection legislation and could thus be used by anyone.

Cover image: www.ingimage.com

Publisher:
LAP LAMBERT Academic Publishing
is a trademark of
International Book Market Service Ltd., member of OmniScriptum Publishing Group
17 Meldrum Street, Beau Bassin 71504, Mauritius

Printed at: see last page

Copyright © Viktor Bazhenov, Anatolii Perelmuter, Yuriii Vorona
Copyright © 2017 International Book Market Service Ltd., member of OmniScriptum Publishing Group
All rights reserved. Beau Bassin 2017
Bazhenov V.A.,
Perelmutter A.V.,
Vorona Y.V.

STRUCTURAL MECHANICS AND THEORY OF STRUCTURES.

HISTORY ESSAYS
Foreword
Respect for the past — that is the difference between enlightenment and savagery.

A.S. Pushkin

That is well to invent yourself, but is not it less than creations to be present or at least to know and appreciate something made by others.

Johann Wolfgang von Goethe

I would like to take part, to be present, as minimum, but if not — to know, at least, how it was.

S.I. Zukhovytskyy
The history of mechanics as the independent science has been existing for about two centuries, and the well-known Goethe’s aphorism: “The history of science is what this science is itself” refers to it.

As to earlier works, one can refer to introductory chapters of J.-L. Lagrange concerning the principles of balance and dynamics in his *Analytical Mechanics* [Lagrange, 1950] and to corresponding chapters in general works in the history of mathematics (Kaestner, Montucla, et al.) [Vakulenko, Mikhailov, 2000].

In the second half of the 19th century mechanics attracted attention of broad circles of naturalists by the discovered energy conservation law and by the attempt to develop in this connection a single mechanistic description of the world. In Russia D.K. Bobilyov published in 1892 a brief historical essay on discovering the basic principles and general laws of theoretical mechanics.

*The History of the Theory of Elasticity and Strength of Material from Galillei to Lord Kelvin* prepared by K. Pearson by the manuscript of A. Todhunter was issued in 1886-1893. A direct translation of most works in the chosen thematic for more than two centuries is presented on 2200 pages [Todhunter, Pearson, 1960]. The work is still preserves its reference value.

A new trend in the history of mechanics was laid in the early 19th century by P. Duhem, who had revealed the medieval mechanics for current science [Duhem, 1903, 1905]. Some scientists consider Duhem’s research as the beginning of modern historical science


A number of monographs were published in the second half of the 20th century in the USA. Two big works by R. Dugas *The History

C. Truesdell, a prominent American scientist and a founder of the current history of mechanics, started publishing his historical investigations in the 50’s. In particular, he was the author of fundamental works in the history of general principles and methods of mechanics of continua of the 18th and early 19th centuries, that were based on the thorough analysis of origins.

The above essays do justice to the history of appearance of general principles and ideas and concern the field of mechanics called the structural mechanics in a broad sense.

There are few books dedicated to the history of structural mechanics; the works by V.L. Kirpichov [Kirpichov, 1903, 1933], S.P. Timoshenko [Timoshenko, 1957], S.A. Bernstein [Bernstein, 1957], C. Truesdell [Truesdell, 2002], [Truesdell, 1968], V.I. Feodosyev [Feodosyev, 1975], K.E. Kurrer [Kurrer, 2008] were most influential for their authors. We should also note interesting historical remarks stated in classical courses of structural mechanics by I.M. Rabinovich [Rabinovich, 1950, 1954].

Except for the book by E.K. Kurrer, all these works were written rather long ago and do not always reflect the important elements of the history of science of the second half of the 20th and early 21st centuries. And it does not concern any concrete results. A more thorough current comprehension of numerous already known ideas is more important now. That is one of the reasons for writing the given essays.

In the authors’ opinion the study and assimilation of the cardinal aspects of theory is very important in the present conditions of computer design that allows an engineer both to effectively use the software and to estimate the results obtained that depends directly on understanding the principles the software is based on. Since according to K. Weierstrass, “the end purpose, which should be kept in mind, consists in understanding the fundamentals”.
The correlation between knowledge and understanding, which comes only when the phenomenon is considered extremely important both in all aspects and in the process of its historical formation.

The well-known repeatedly published book by S.P. Timoshenko [Timoshenko, 1957] was written as based on the lectures in the history of material strength, which he had been reading for 25 years; and his teacher V.L.Kirpichov headed for many years a scientific circle, where he held discussions on mechanics assumed as a basis of his book *Discussions on Mechanics* [Kirpichov, 1933].

V.L.Kirpichov and S.P. Timoshenko played a great part in organizing education and science in Ukraine. V.L.Kirpichov was a founder and first director of the leading higher educational institutions in Ukraine: Kharkiv Technologic Institute and Kyiv Polytechnic Institute. S.P. Timoshenko was one of the founders of the Ukrainian Academy of Sciences in 1918. On his initiative technical sciences were ranked as academic ones, it was for the first time and a Subdivision of Applied Natural Science was organized in the Academy structure.

Returning to the fundamental book by S.P. Timoshenko [Timoshenko, 1957], the authors have to say that when they planned this work, the availability of such an outstanding precursor had played the important and ambiguous role. On the one hand, we wanted to supplement the study and to make it longer in time; on the other hand, we were not sure whether we could keep to the level set by S.P. Timoshenko. Thus the authors have not dared to repeat such a broad approach to the problem and have limited themselves to some essays.

Thematics of the essays given below and dedicated to selected questions of structural mechanics and the theory of structures does not pretend to all-embracing presentation of their history, and doubtlessly it is fragmentary. As to authors’ intention these essays may be read as independent ones, but since some their elements intersect, we could not avoid some repetitions. As regards the thematic choice, it was determined by the authors’ interests, and it is
open to extension and does not pretend to ranking problems as to their importance.

A reader, who wants to extend thematics of our essays, may search in Internet, where the authors also took some facts from the history of structural mechanics. And in this case we will think of our work as of the useful one.

July 2016 Authors

References

Bermsthein, S.A. (1957), Ocherki po istorii stroitelnoi mekhaniki [Essays in the history of structural mechanica], Gosstroizdat, Moscow, Russia.


Kirpichov, V.L. (1933), Besedy o mekhanike [Discussions on mechanics], 2nd edition, Gostekhteorizdat, Moscow, USSR.

Klein, F. (1937), Lektsii o razvitii matematiki v XIX stoletii [Lectures on development of mathematics in the 19th century], ONTI, Moscow-Leningrad, USSR.

Lagrange, J. (1950), Analiticheskaya mekhanika [Analytical mechanics], vol. 1 and Vol 2, GITTL, Moscow-Leningrad, USSR.

Makh, E. (1909), Mekhanika [Mechanics], St-Petersburg.


Newton, I. (1989), Matematische nachala naturalnoi filosofii [Mathematical principles of natural philosophy], Translated from Latin by A.N. Krylov, Nauka, Moscow, Russian Empire.

neopredelimye sistemy [Course of structural mechanics of rod systems. Vol. 1. Statically determinable systems. Vol. 2. Statically indeterminable systems], Gosudarstvennoye izdatelstvo po stroitelstvu i arkhitekturo, Moscow, USSR.

Timoshenko, S.P. (1957), *Istoriya nauki o soprotivlenii materialov s kratkimi svedeniym [The history of science of material strength with brief data from the history of theory of elasticity and theory of structures], Gospeirizdat, Moscow, USSR.


Feodosyev, V.I. (1975), Desyat leksiy besed po soprotivleniyu materialov (ten discussions on material strength), 2nd edition, Nauka. Moscow, USSR.


Duhem, P. (1903), Les Origines de la statique.

Galileo Galilei (1638), Discorsi e dimostrazioni matematiche intorno a due nuove scienze, Leida.

Kurrer, K.-E. (2008), The history of the theory of structures, Ernst & Sohn Verlag für Architektur und technische Wissenschaften GmbH & Co. KG, Berlin, Germany.


Essay 1

STRUCTURAL MECHANICS AND NOT ONLY:
DEVELOPMENT OF REQUIREMENTS TO NO-FAILURE OF
STRUCTURES
Each country and each city – and, properly speaking, each person – decide themselves which level of safety is admissible. But it seems that some designers forget sometimes that the so-called safety factor can protect from unskillfulness and somebody’s mistakes, but not from their own.

R. Byrne
Introduction

Provision of the strength and resistance of building structures is one of the most important tasks of structural mechanics, its performance consisting of at least two parts:

- to analyze the structure behavior, to determine its reaction to external effects, to estimate external conditions and everything that structural mechanics is concerned with as science and that is written in manuals in structural mechanics;

- to develop and substantiate the rules concerning inadmissibility of structure failure and provision of structural reliability, that is everything performed with the help of analysis rules presented in building codes (technical regulations).

In the world of current computer design the safety margin is rather amazing: loads and parameters of material strength are set with accuracy of 10-20%, force calculated with accuracy of 0.01%, and safety margins achieving 50-100% are concealed sometimes in certain specifications. In recent times the safety margin was a simple number included in a normative document, but modern standards contain complex systems of partial safety factors (margins of strength), which do not always allow estimating the real no-failure reserve. What is science and what is the volitional decision (if not the black magic) here? What values of safety factors were used and how have they changed in the course of time? The essay considers how the home and foreign design specifications changed, what safety margins they provided, what was the procedure of the exchange of ideas and borrowing of methods, and how, at last, the approaches to standardization target changed themselves.

A contemporary engineer thinks most often that the accurate keeping to design standards is a necessary and sufficient condition for providing no-failure of created structures. But analysis shows that a lot of old buildings that do not meet current requirements, still exist
and are in use. So, are our requirements absolutely necessary? The
answer is probably negative, but there arises a question: how our
precursors met the no-failure requirements. This problem is difficult
to understand without knowledge in history.

The history of development of home and foreign design
standards, as any history (if it is not presented tendentially) gives not
only actual knowledge of the past experience, but permits, to some
extent, predicting tendencies of the problem development. In the
authors’ opinion the current standards are on the threshold of a new
stage. The approaches to standardization, which function in Ukraine
and other countries, do not almost allow for the fact that the up-to-
date design is created with the help of computer analysis, and this is
already a tendency to improving the standards.

1.1. Prehistory

As centuries-old construction experience shows, the problem of
the structure strength and safety existed at all times, and it is urgent
now. The philosophy of safety of designed buildings and structures
developed by certain stages, and in its main course this development
proceeded under the slogan of more detailed prediction of structure
service, study of the nature of loads affecting these structures, more
expressive description of requirements to structural form and
conditions of meeting such requirements.

The history of construction evidences that even in the most
perfect old structures one can find blunders which reveal ignorance
of the principles of strength of materials and theory of structures.
Superstitious fear of the unknown secret of material made builders to
beg assistance of the other forces using prayers (that still continues),
charm exorcism, and even sacrifices. The profession of a builder was
always considered as that of great responsibility, and possible
construction mistakes of builders could have grave consequences for
those who made them.
Norms ensuring structural safety were, as a rule, too illegible. It is considered that the earliest known building codes and rules were rules included in Hammurapi Code, dated from 1772 B.C. Those were as follows:

- If a person builds a house for somebody, and does not construct it as required and the house built by him would fall and kill its owner, this builder is to be executed.
- The owner’s son being killed, the builder’s son is to be executed.
- The owner’s slave being killed, the builder has to pay for the slave.
- The owner’s property being destroyed, the builder has to pay for everything destroyed, and, since he did not build in the required way, and the house built by him has fallen, he has to build it again for his own expense.
- If a person builds a house for anybody, and he has not finished his work, but walls are ruined, the builder has to restore the walls for his own expense.

If a person has shown carelessness, when strengthening a dam on his land, he has to recover losses up to selling him into slavery. A certain building code may be found even in the Bible (Second law, chapter 22, verse 8): “When you build a new house, make handrails for your roof, and you will not call blood to your home, whether anybody will fall from it”.

The regulations developed by practice for performing some building works existed, for example, in ancient Rome. It especially concerned the road engineering, that was principally important for the Empire which occupied great territories. Some roads constructed at that time are still in use. The roads of Ancient Rome were constructed according to first specifications, the so-called 12 tables developed even in 450 B.C.
Total road thickness in Rome was 80-130 cm, though some of them reached 240 cm. As a rule, the roads were multilayer (4-5 layers) (Fig. 1.1). The lower layer serves as the base of 20-30 cm thick stone plates, laid on the roadbed compacted through solved covering, with further their leveling with sand. The second 23 cm thick layer consisted of concrete (broken stone put into mortar). The third 23 cm thick layer also consisted of fine gravel concrete. The last upper layer of the road was covered with great stone blocks of 0.6-0.9 m² space and about 13 cm thick.

Successful decisions in engineering, which had been checked by practice, repeated¹, they were elucidated in detail, for example, in the treatise by Vitruvius *Ten Books in Architecture* (the year 13 B.C.), the only preserved ancient book in architecture [Vitruvius, 1936]. The author generalized the experience of Greek and Roman building art, considered a set of concurrent town planning and engineering problems, practical building rules and principles of artistic perception. As a result the treatise has become the encyclopedia of technical knowledge of that time.

The first code of requirements (*The Building Statute*) appeared in the Kyiv Rus territory in the 11th century under Yaroslav the Wise. It determined locality and materials fit for construction, altitude of

¹ Maybe hence originates the tradition of shameless copying that is still preserved in architecture and condemned in other arts.
buildings, included recommendations which concerned the arrangement of premises in buildings.

These and other directions, rules, traditions, etc., had the aim to prevent (to the extent of the then understanding) the destruction of buildings or to make that their inevitable demolition could be compensated by repair.

1.2. First investigations

There are two types of destruction, when renewal or restoration rarely has any sense – the destruction by fire and fall of the structure. That is why these two forms of building destruction were the objects of certain precautionary measures. Special certified measures concerning the fire safety of buildings existed even in Ancient Rome; they appeared in London in the 17th century, when the Law composed by Sir Matthew Hale had regulated the city renovation after the great fire of 1666.

It was much more difficult as to ensuring strength, and thus, the corresponding regulations appeared much later. It was impossible to determine requirements to design of various structures until it was theoretically grounded. It began with the works by Galilei published in 1638 after his famous Dialogue and Mathematical Proofs Concerning the Two New Fields of Science [Galilei, 1964]. The science of strength was one of those fields.

As S.O. Bernstein wrote [Bernstein, 1957]:

“It may look strange that this question was never raised before Gallilei for long centuries of the human culture, though wonderful pieces of architecture and bridge building from the Antiquity to Middle Ages have reached us. Nevertheless the attempts to find pre-Gallilei works about strength go unrewarded.

That was only Leonardo-da-Vinci, this universal genius, who studied everything (and has not finished any study), was engaged in the problem of strength and resistance before Gallilei, but his work
remained unpublished and thus had no influence on development of the science of strength”

There is almost no science which founder might be called. In most cases a new branch of science grows gradually from the old bole. No wonder: to create a new science one should see new paths, find something new – only great scientists can do it. Gallilei has said a new word: he was the first in the history of humanity to raise a question of the body strength and the first who tried to settle it.

Gallilei considered the body strength in the moment of rupture (now, in limiting state). He was not interested what was the body path and what stages it had passed to reach this state. Gallilei’s approach was taken without objections, and all scanty experiments of that time (and that was the only method of investigation then) were made with the only objective – to find the breaking load value and breaking mode.

The lack of experimental data concerning the structure behavior at high loads made to resort to more-or-less probable hypotheses about the scheme of breaking, thus introducing elements of conditionality and self-will.

That may be seen especially distinctly in the history of arches design. The work by Coulomb [Coulomb, 1773] may be considered typical in this field. He postulated four possible schemes of the arch break-down (Fig. 1, 2) and found four boundary values for thrust: two turn-over boundaries $H_{v}^{\text{max}}, H_{v}^{\text{min}}$, and two shift boundaries $H_{s}^{\text{max}}, H_{s}^{\text{min}}$. If the both lower boundaries are less than both upper ones, for example, if $H_{s}^{\text{min}} < H_{v}^{\text{min}} < H_{s}^{\text{max}} < H_{v}^{\text{max}}$, then the vault balance is possible, the real thrust being distributed between the inner members of this series of inequalities; if one of lower boundaries is bigger than one of the upper ones, the balance is impossible.
Knowledge of change limits or even exact value of the breaking load gave no answer to the questions how one can use this knowledge in practice, and practical builders had to solve this riddle to their mind.

![Four arch breaking schemes by Coulomb](image)

And they, like their ancestors, settled this question by intuition, by the trial-and-error method, they learnt at the lessons of damages and falls of structures. Each accident enriched builders with new knowledge, posed new tasks. If knowledge was not enough, they introduced the assurance factor (they do it even now) in the engineering design. Since nobody knew which unforeseen, unknown phenomena were allowed for and what was this factor – it was practically the ignorance factor.

1.3. Admissible stress

Theoretical grounds for structures design had been formulated in the methods of structural mechanics, which became an independent scientific discipline by the end of the 19th century. It had been found that science of the 18th century with its speculative methods of searching the limiting state could not solve the simplest problem – to make analysis of a beam for bending, and it could not do it because could not answer the question: how a beam behaves under the load which it has to bear?

The antiquated approach had been radically broken by the works of C.-L. Navier, who took the road of studying real
service of a structure under load, its design in the working order.

The difference between the old (Gallilei) and new (Navier) approaches was as follows:

- A principle of the limiting or ultimate state follows from the pattern of probable structure break-down and determines the value of the load under which such a destruction may occur. The admissible load is determined dividing the breaking load by the safety factor.

- A principle of serviceable or initial state defines the stress-and-strain state of the structure under real workloads, taking that the limiting state is fully similar to serviceable one in such a way that the ratio of loads, forces, stresses and displacements in the both states is the same and equals the safety margin.

Under this approach it will suffice to study the serviceable state, i.e. the stress and displacement under the design load and to find their relation to the breaking load.

Since in so doing the limiting stress value for given material which, when divided by the safety margin gives the so-called admissible stress, is considered known from experience, the whole design is reduced to the comparison of real serviceable stresses with admissible stresses. For this reason the design in serviceable condition is often called the design by admissible stress.

Only one question remains to be answered: what this admissible stress would be?

W.J. Rankine, a well-known Scottish engineer, physicist and mechanician, has determined the safety factor as a ratio of material ultimate strength to maximum admissible stress under the effect of real or serviceable load on the structure. Rankine has also indicated the difference between the permanent load that may be determined exactly and temporary load which value cannot be established with the same accuracy.

He thought that $k=4.0$ is an admissible value of the safety factor.
This admissible workload was included in building codes for different materials and building structures by the late 19th-early 20th centuries.

Admissible stresses accepted in various countries were essentially different. Thus in England the admissible forces for structural steel were based on the 4-fold safety factor in respect of the medium yield limit which equaled about 432-494 H / mm². The law of 1909 of the London County Council defined the admissible bending, tensile and compressive stress as 116 H / mm². Russian Setting Proposition [de Rochefort, 1910] for the structures of welding iron established the admissible tensile stresses equal to about 80 H/mm2 and compressive stresses of 65 H / mm². In Germany these values were 115 and 95 H / mm², respectively.

Later values of admissible stresses were repeatedly revised and those for steel structures corresponded to the 2-fold assurance factor up to the 40’s of the 20th century, but rather in respect of the yield point sorting minimum $\sigma_r$, than of its medium value.

In the Soviet Union the value $\sigma_r$, for Ct3 steel was taken as 240 H/mm². The assurance factor allowed for a lot of factors taking unfavorable effect on the structure behavior and depended, in particular, on the number and character of loads for the structure design. Up to 1942 the highest assurance factor $k = 1.7$ and the lowest admissible stress $[\sigma] = 140 \text{ H / mm}^2$ were taken under the design of loads which were permanent or coinciding, for example, permanent load and snow. When allowing for a higher quantity and randomness of loads (wind of hurricane intensity, the temperature influence) the admissible load was taken as $[\sigma] = 170 \text{ H / mm}^2$, and $k = 1.4$.

A planned decrease of the assurance factors based on the stored experience had been corrected by conditions of materials deficiency during the World War II. In the Soviet Union in 1942 the values of admissible stresses for steel structures were raised to 160 and 180 H / mm² according wartime standards, the assurance factors were taken as 1/5 and 1.33 H / mm², respectively. The variable assurance
factors, being calculated by the break-down stage and functionally depending on the ratio of temporary loads to permanent ones, were adopted for reinforced concrete structures [U 37-42].

Similar changes were made in Great Britain, where the assurance factor for steel beams and columns equaled 1.8 before 1939, and during the World War II (1939-45) it was decreased for the steel beams to 1.6\(^2\). After the war in 1948 this factor was increased to 1.65-1.7. The curves of admissible stresses for columns were founded on the assurance factor of 2.0 up to 1964, when it was lowered to 1.7.

It should be noted that in the case of buckling the assurance factor was changed using a more refined method. To obtain the admissible stress for the bar with centric compression an assurance factor is introduced in addition to theoretical value of the critical stress; this factor may be presented in the form of the product of the ordinary assurance factor \(k\), used under tension, and a special factor \(k_1\) that allows for a decrease of carrying capacity at the expense of random eccentricities. It appeared that short columns have small defects, and their residual saggings are small, small residual saggings are also characteristic of very slender columns with springy reaction to random effects. That is why, on recommendation of N.S. Striletsky, the assurance factor \(k_1 = 1\) is taken for slenderness \(\lambda = 0\), \(k_1 = 1.4\) for slenderness \(\lambda = 90\) and \(k_1 = 1.15\) for slenderness \(\lambda = 200\), that gives at \(k = 1.5\) the assurance factors equal to 1.5; 2.1 and 1.725, respectively (see extra bold points on Fig. 1.3).

\(^2\) The alteration was made as amendment to British standard BS 449 [Revision] “as extraordinary measure in expectation for more detailed consideration of the problem”. It was said: ” the permission for increasing stress is to be terminated after the completion of the 6-month period after the war, if the amendment will not be approved as a constant change of the standard following a normal procedure”.

20
Fig. 1.3. Admissible stresses for a compressed column according to NiTu 121-55

Similar considerations were present under the development of the Great Britain Standards [Alasdair, 2011]. The curves of admissible stresses from British Standards of different years shown on Fig. 1.4 demonstrate disproportionate changes in columns with different slenderness.

Fig. 1.4. Admissible stresses for a compressed column according to BS449
1.4. Breaking load

The analysis of serviceable condition foresees that the highest stresses, which appear under the design load, are compared with admissible stresses for the given material. This formulation includes two propositions to be specified, that is: which stresses are to be compared (a), and how can we establish the admissible stress value (b).

The answer to the first question requires accepting a certain strength criterion, and that is the first supposition (in some cases formulation of such a supposition verges on the high-handedness).

The decisive stress being designated as \( \max R \) and its admissible value under a simple tension as \( [R_p] \), the analysis of serviceable condition for any strength criterion leads to inequality.

\[
\max R \leq [R_p]
\]

If some strength criterion is chosen and thus the left part of the limit inequality is established, a question remains to be answered: how can we define the admissible stress, i.e. how can we find the right part of this inequality. An assumption was made in the explicit form by Saint-Venant in the 1980’s that for simple tension or compression one can restrict himself to determination of admissible stress.

There is one more admission used in an inexplicit form: when the decisive stress reaches the limit value even in a single point of the structure, that means it reaches the limit condition for all the structure. It is easy to understand that this condition is fulfilled only for absolutely brittle material which follows Hook’s law up to the break-down. Thus the third admission may be called the brittleness condition. It proved to be most vulnerable.

The brittleness condition also came into conflict with experience even in the 19th century. One of the first signals of its inaccuracy was the result of the attempt to specify the truss design allowing for
rigidity of nodes. It appeared that the effect of nodes rigidity results in a considerable increase of stresses in the truss columns as a result of their bending compared to design values. The latter are obtained in the ordinary way, using a hypothesis of hinged nodes, though, as it follows from experience, the results of “node design” lead to entirely reliable structures. The next series of blows was delivered as a result of studying the behavior of structures of plastic materials under loads close to limit ones. It appeared in numerous experiments that the behavior of statically indeterminate structures change qualitatively after leaving purely elastic area.

The matter of all these phenomena completely comprehended even in the early 20th century was as follows: the equality of safety factors does not follow from the equality of the highest stresses, or otherwise, similarity between the operation and limit condition is possible only for ideally elastic perfectly brittle material. There arose a question of analysis of the structure limit condition, its behavior in the moment of breaking. This concerned the study of plastic deformations and analysis of the effects of geometrical nonlinearity in particular.

The matter of the problem for geometrically nonlinear resilient structure, e.g. a flexible plate of the vessel plating was distinctly stated by I.G. Bubnov even in 1908. He has shown that introducing the assurance factor in the load value or in the stress value, we obtain different results, and he has clearly substantiated a necessity of the first approach [Bubnov, 1908].

As to physical nonlinearity, the question of applying the theory of plasticity to behavior of restrained beams after the stage of elastic deformation was first raised by Gabor Kazinczy [Kazinczy, 1928], who had shown that the theory of plasticity allows achieving approximately 25% material economy for a beam with clamped ends. It is suffice to attract attention to the problem.

Reinforced concrete structures which were used on a mass scale in the first quarter of the 20th century caused the highest anxiety. In the late 19th century the reinforced concrete structures were
designed according to admissible stresses using laws of behavior of elastic materials.

But already in 1904 A.F. Loleit presented the work *On Strength Coefficient of Reinforced concrete Structures* [Loleit, 1904] where he had shown that the design of reinforced concrete members under bending in the resilient stage of their service is absolutely inadmissible. He wrote: “critical loading corresponding to momentary equilibrium which directly anticipates the breaking … allows determining the safety factor with accuracy meeting the most severe requirements of practice”.

Arthur Loleit became the initiator of including the method of breaking loads in the design standards of reinforced concrete structures. In 1928 the Commission of Building at the Council of Labor and Defense charged him to make a draft of new specifications and standards on the basis of comments of the initial project made by the Bureau of Standards of Gosplan of the USSR; the comments were received from various departments and institutions at different times.

The project was published in 1929 under the title *Specifications and Standards of Design and Erection of Concrete and Reinforced Concrete Structures*, and sent for reviewing to specialists and organizations; and in April 1930 it was submitted to discussion at the First All-Union Conference on Concrete and Reinforced Concrete in Moscow. A.F. Loleit made a report *New Project of Standards* [Loleit, 1930]. Since there also existed competing propositions which, to tell the truth, rather concerned numerous details than the major thought of using the breaking loads; it was decided to perform check experiments. [Loleit, 1934]. The results of experiments allowed including Loleit’s approach in the standards that was made some years later (OCT 90003-38).
The approval of the method of design according to breaking forces as a law meant not only preference of one method compared with others. That was not just a choice of the method of design which gives the most exact results. That meant giving preference to scientific experiment as a basis of development of the theory that had far-reaching consequences.

Charles S. Whitney, American engineer, upheld the ideas similar to those formulated by A.F. Loleit. He expressed them beginning from 1926 [Whitney, 1929], and in 1937 Whitney published a detailed Project of Standards based on the method of design according to breaking loads; in 1956 the project was introduced (with some modifications) in the Building Standards of the American Institute of Concrete as the optimal method of design. In 1955 Whitney was elected President of the above Institute, and that could favor the decision.

That was followed by revision of general approaches to the method of breaking loads. In structural mechanics a continuous beam is a certain touchstone for any new method of design of statically indeterminate systems.

That is why that new ideas of design, which aroused from consideration of plastic deformations of material, were applied to this kind of a beam. It follows from the theoretical analysis that the real safety margin in the continuous beam is much higher than it follows from the ordinary “elastic” analysis. The experimental check has shown that the real behavior of the continuous beam something differs from this theoretical design, since the moments are not placed in equilibrium in the most stressed intersections as a result of strengthening of material which passed the yield site. Owing to this fact the real safety margin is a bit higher than that which follows from the theoretical design.
However the transfer from the limit stress to breaking force has not changed the basic paradigm of the design according to admissible stress. It was the breaking load of reinforced concrete member that was divided by the safety factor instead of the limit stress of material.

It should be noted that other definitions of the safety factor are also used in the design practice. For example, the theory of strength of Mohr-Coulomb is widely used in soil mechanics; according to this theory no break-down takes place, if the following ratio
\[ \tau \leq \sigma \cdot \tan \varphi + c \]
is fulfilled, where \( \tau \) – tangential, \( \sigma \) – normal stress that affects the considered site; \( \varphi \) – angle of internal friction; \( c \) – specific adhesion. The safety factor is sometimes determined as a value by which the soil resistance parameters \( \tan \varphi \) and \( c \) should be divided, for the Mohr-Coulomb strength condition turned into equality. Such an approach is substantiated by the fact that in contrast to most structures, where load variability is much higher than resistance variability, the soil design characteristics are most variable factors of soil bases. So, when estimating the safety factor, it will be more logic to vary resistance than load.

Determining the design loads was another problem of design with the use of safety factor. Permanent load could be calculated according the dimensions of structures and known mass of materials. Temporary load estimation was first rather approximate. One could determine loading by the crowd of people which (as it was thought in England, where such measurements were made for the first time [Adams, 1894]) is 150 pounds per 1 foot\(^2\) (\( \approx 700 \) kg/m\(^2\)). However, such loading is possible in conditions of panic, and the safety factor has to allow in part for this circumstance. That is why the temporary loading from 40 to 80 pounds per 1 foot\(^2\) (from 200 to 400 kg/m\(^2\)) was considered as admissible for buildings, except for storehouses, where the loading may be higher,
Mitchell, whose studies were used 20 years later, could reach more exact calculation of the effect of maximum accumulation of people, since he observed their quantity in the dinner time, in free days of the week, during a sale or directly before holidays, depending on shop specialization and location. That gave the researcher the load value of 50 pounds per 1 foot\(^2\) for the crowd at the upper part of staircases and at shop exits. Thus, more accurate studies showed a tendency to lowering the values of maximum loads.

During the 19\(^{th}\) century the columns were always designed for full temporary loading from all the stories they supported. After 1900 the local building standards of Chicago admitted a decrease of temporary loading, since it was considered doubtful that the temporary loading could be concurrently maximum at all stores. The standards foresaw the account for maximum temporary loading of the upper story structures only, 95\% load of the next to last story, 90\% – of allowed one, and so on, until the temporary load reached 50\% which were taken for all the lower stories. Similar recommendations were adopted in the London Building Regulations in 1909; they are still used in most building codes.

The problem of combining loads is also of importance in connection with allowance for winter load. First examples of the account for the wind load were founded by analogy with regulations of railway bridges design adopted after the well known fall of the Tai bridge in Scotland in 1879 under the hurricane wind [Perelmuter, 2011]. It was considered that a bridge can hardly bear a maximum temporary load, when high horizontal loads appear under the storm wind. Thus, according to new regulations the normal stresses could be increased by 20\% in the case of combination of the vertical and wind loads. This regulation was introduced in practice of design of multistory buildings and included in American and English building codes, though its validity still remains open to question.

In the 20’s of the 20\(^{th}\) century a new branch of industry – the aircraft construction stimulated the search for more precise bases of the
assurance factor. Its reasonable decrease which causes no danger for the structure reliability and, naturally, favors economy of materials and means. That is of importance for civil engineering and especially for aircraft construction; too heavy aircraft cannot fly up. Thus the assurance factor in aircraft structure is much lower than in buildings, but this lower factor is combined with a thorough control of material quality, frequent check of technical condition of structures, renewal of damaged parts, precision of the structure design methods based on mathematical analysis. The methods of design developed for the aircraft industry had a certain influence on the design of buildings.

It should be noted that a somewhat modified method of admissible stresses still remains in the air-space branch. For example, in the USA [Muller & Schmid] the assurance factor \( k = 2.0 \) was used in respect of the maximum probable load on military aircraft up to the early 1920’s. In March 1934 Heads in aircraft design (HIAD) established the value of assurance factor \( k = 1.5 \) as a demand to design of a fuselage and wings.

It was recognized that the loads, exceeding the limit ones, may appear during a flight. Then followed a required mandatory examination and renewal, before giving permission for further flights.

In so doing a minimum possible thickness (as to tolerance) of the parts or nominal thickness divided by 1.1 were to be used for determining stresses in pressure vessels and for determining critical stresses.

As to mechanical characteristics of materials, it is foreseen that providers have defined them using some statistical criteria. Thus, mechanical properties become completely isolated from determining the assurance factor.

The document NASA-STD-5001B, *Structural Design and Test Factors of Safety of the Space Flight Hardware* has established the value \( k = 1.4 \) for the design of space objects. But this factor concerns only examined and tested parts of the structure. The assurance factor
k =2.0 was established for non-tested parts of the structure that had to resist insufficiently exact calculation or inefficient quality control.

1.5. New ideas

Thus the approach based on the assurance factor which was established on the basis of engineering intuition, design experience and structure service was used in structural analysis from the 19th century to the 50’s of the 20th century.

In the meantime propositions concerning the development of the methods of structural analysis were substantiated in a number of works. In 1926 M. Maier offered using the probability methods to choose parameter values introduced in the analysis instead of the permissible stress design. [Maier, 1926]. It should be noted that though Maier was the first to publish a book on the probabilistic method in 1929, but it was Gábor Kazincky, who was a pioneer here and defended the use of he probability methods even in 1913, though his work was published only in 1929.

In 1929 M.F. Khotsialov, allowing for variability of the basic parameters, offered structural design proceeding from a certain regulated probability of their crash [Khotsialov, 1929].

But M.F. Khotsialov’s formulation “to design with allowance for accident possibility” met a strong repulse of traditionally disposed engineering community; its representatives were fully confident that the structures made with a good assurance factor were absolutely reliable. Khotsialov’s ideas were rejected for a long time. In so doing the engineering circles did not understand that he had proposed just the approach to determining “a good assurance factor”.
The ideas of M. Maier and M.F. Khotsialov were essentially developed in the works by M.S. Striletsky. M.S. Striletsky correctly reasoned that the problem of the assurance factor was exceptionally important and thus, he has devoted about 15 works to it (see, for example [Streletsy, 1935,1936-1, 1936-2,1947]).

Until a design on the basis of probabilistic approach appeared in engineering practice it was considered that the assurance factor is a special number endowed with some exceptional properties. It was foreseen that following it ensures the structure reliability, while even its inconsiderable decrease causes danger for the structure.

But as to compilers of specifications the assurance factor always remained for them a generalized presentation of precautionary measures providing a satisfactory level of safety.

In due course of time the assurance factor became more and more differentiated for various types of structures. For example, load distribution in a motor-road bridge depends on cars arrangement in different points across the roadway width, while in the railway bridge the load is distributed only at the places of rails. The design load on the railway bridge is close to real, since a train always moves along the estimated line that does not take place as a vehicle moves along the motor-road bridge. Thus, the assurance factor in the design of the motor-road bridge is considerably higher than in the railway one.

Under these conditions the permissible stress design or ultimate load design of structures deals with a single assurance factor independent of operation conditions of structural members and almost independent of the kinds of loads. When analyzing this problem in the 40’s, M.S. Striletsky formulated distinctly the idea
(though it was inexplicitly used in the works of other authors) of separate analysis of load variability and material strength and of their separate account in specifications, the idea which was assumed as the basis of the method of limit-state analysis.

Aviation engineers were probably the first to understand changes in the philosophy of strength analysis, when random values are under consideration, and failure probability should be allowed for. The old definition of the safety margin loses its simple value – the failure probability depends now on the form of load \( P_Q(s) \) and strength \( P_R(s) \) probability distribution. In 1942 A. Pugsley [Pugsley, 1942] demonstrated how a double integral over load and strength distribution may provide for the probability of failure and derived a formula for the failure probability \( P_f \)

\[
P_f = \int_0^\infty P_Q(s) \int_0^\infty P_R(s) \cdot ds \cdot ds.
\]

Pugsley used hypothetic distributions, since available actual data were not sufficiently precise especially in the limiting value area, but he could demonstrate the failure probability sensitivity to changes in adopted distributions.

In the field of building structures the probabilistic nature of the problem was studied intensively in the pioneer works by M.S. Striletsky himself [Streletsky, 1935, 1936-1, 1936-2, 1947], A.M. Freudental, [Freudental, 1938, 1947], M Plot [Plot, 1936], and V. Wierzbicki [Wierzbicki, 1936], where not only material strength parameters, but also load parameters were used as random values. Under these conditions they did not write about failure as a random event which probability should be changed in some way. Even in the simplest variant, which was used in the above works, the probabilistic method of analysis was not accessible for the direct use. The question was not only in the
absence of the necessary statistical information, but also in complete break with the previous experience. So, a half-probabilistic method of limit-state analysis devoid, in principle, of the above disadvantages and such that realizes a practical variant of the use of some points of probabilistic approach is a substantial stage in development of structural analysis. Being determined in its form this method is based on using the methods of statistical analysis, when finding reliability indices characteristic of the method. A conditional system of the above reliability indices was offered in 1945 by I.I. Goldenblat, M.G. Kostyukovsky and O.N. Popov and assumed as a basis of the design model for development of building codes and regulations [Baldin, et al., 1951]. This work was made by the commission which included V.O. Baldin, O.O.Gvozdyov, I.I. Goldenblat, Yu.M. Ivanov, V.M. Keldysh, L.I.Onishchyk, M.S. Striletsky and K.E. Tal.

The method of limit-state analysis was introduced in the USSR as a key principle of building structural analysis on January 1, 1955 jointly with the approval of the first edition of the State Building Code and Specifications. Then the limit-state analysis was recognized all over the world, and now it is taken as a basis of most international and national design standards, in the Eurocode system in particular, where it is called Method of Partial Reliability Index [EN 1990].

It is known that the introduction of the method of limit-state analysis beyond the limits of the USSR lasted tens of years. It was Brinch Hansen, who might be called a pioneer of this process; on his proposition this method has been used since 1956 for settling the problems of geotechnologic engineering in Denmark. This initiative was taken up in the USA, e.g. the method slowly laid its way under the title Load and Resistance Factor Design (LRFD).

First application of some LRFD ideas may be noted in the American Concrete Institute (ACI), where in 1956 the ACI Committee 318 adopted the Building Code and Regulations Requirements for Reinforced concrete (ACI 1956). The document
was laconic, the design method was titled *The Boundary Force Design*. These codes did not include the notion of resistance coefficient, thus all the margins of strength were included in the load factor. But these coefficients were different for different types of load as well as for different combinations of load. In the next version (ACI 1963) they used a complete format of LRFD, including the resistance coefficients. The test procedure was still known as *Limit Force Design*, but was already identical to LRFD regulations. However, both loads and resistance coefficients in the ACI code were not based on rational analysis, but defined by intuition and thought of participants of the Standardization Committee.

In 1969 Cornell [Cornell, 1969] published a paper in the journal of the American Concrete Institute, where he offered a conception of standards which rested on the probability basis. And only in 1980 Ellingwood with co-authors [Ellingwood et al., 1980] presented information to the USA National Standard Bureau about load reliability indices for buildings based on the probabilistic analysis.

Was the above delay explained by unfamiliarity with the works of soviet scientists or by other reasons? The lack of necessary statistical information in the mid-50’s for substantiating the values of partial resistance indices could play the leading part.

Authors of this method, who relied upon future studies, were honest, when told that in the very beginning they fitted obtained results to decisions tested by previous experience.

In western countries, where a command system of innovation introduction is not approved, this gave no opportunities to convince the engineering circles of the urgent necessity to change the approach to design, the more so as the fitting to earlier experience gave no considerable economic effect. Besides, some economy had been achieved for the structures mainly affected by the persistent loads with minimum overload factor, that is the 3-
10% economy for principals and footing trusses, while crane girders remained practically unchanged and columns of industrial buildings either remained unchanged or even became a bit heavier.

It should be noted that the idea of heredity is also present in western standards. Thus, in the foreword to the British Standard [CP110] it is stated that *the corresponding statistical data are insufficient to allow the method of partial reliability indices to be developed in full agreement with probability theory, and the used values of indices were based on current practice*. However it is said that the method of partial reliability indices has the advantage, since *in due time making corrections in the Code will become simpler, as far as new knowledge about load and support variability will become accessible*.

The fact that a more detailed analysis, inherent in the methods of partial safety factors, can come into conflict with the method of permissible stress design also created certain difficulties. Thus, when composing Eurocodes, the European Commission for Standardization (CEN) faced the fact that some countries, which used the method of permissible stress design, were rather afraid that some designs based on the limit-state analysis will point to formal violation of requirements of the latter. Other countries indicated that the already erected buildings prove to be less efficient under limit-state recomputation. These contradictions were smoothed by some deviations from statistically justified values of reliability indices. Thus present-day Eurocodes are the result of a compromise, and complete probability of all standard requirements may be expected only in their next editions.

Joint Committee on Structural Safety (JCSS) works intensively over this problem. The Committee was created on the initiative of six international associations in the sphere of construction engineering including International Committee for Reinforced Concrete (CEB), International Council on Research and Innovations in Building (CIB), Reinforced Concrete International Board (FIB), International Association on Design of Bridges, Structures and Erections (IABSE)
and International Union of Laboratories and Experts in Construction Materials, Systems and Structures (RILEM).

In Eurocodes, as well as in other foreign publications, the method of design limit states appears under the name of the method of partial reliability indices. Two names represent the most substantial sides of the method, each of them being independent in a certain sense.

The method being considered from the viewpoint of the use of the limit states, it should be remembered that the method is based on the idea of giving up a detailed analysis of all the states of the structure, except for the limit ones, design requirements to the object being formulated in respect of these states (here we can see full heredity with respect to the method of breaking loads). Such an approach, besides the well-known advantages, has a serious shortcoming, since, taking, e.g. the strength condition as one of the limit states, and the structure being designed in such a way that we could say with a certain degree of sureness that this condition will not be violated during the whole term of service, we can say almost nothing about correspondence of the level of actual stresses to normal (non-limit) state under the conditions which are most frequently realized in operation process.

The structure states, which are most often realized in service conditions, most frequently define its service life. But from the viewpoint of margin analysis both the structure of a dam, which ordinary level of load is not too far from the limit permissible one (e.g. it is 80% of design), and the structure of a chimney, where the appearance of design load is rather rare, and ordinary load constitutes, e.g. 15% of the design load, can be found almost equivalent.

When concentrating attention on the system of partial reliability indices, we will see the substitution of one general assurance factor by the product of several (partial) indices; each of them is connected with a certain side of the problem. The material reliability index $\gamma_m$ and load reliability index $\gamma_f$ have become the basic ones.
This feature of the method of limit states led to considerable activation of the study of the above indices and development of design standards. It is detalization in the use of the combination of partial reliability indices provides (to be more precise, has to provide) the situation of equal probability of the limit state realization of the two above considered objects, their ordinary state sharply differing in the degree of closeness to the limit one.

But there is a certain problem, since we can rely on equal reliability only in respect of those factors (e.g. internal effects), which were taken into account under the design and which statistical characteristics were used under setting the method design factors. And in case of certain occasional perturbations not foreseen by the project (and standards) the probability of exhaustion of a 20% assurance in the first case is much higher, than the exhaustion of 85% assurance in the other case [Perelmuter, 2015]

Besides, one should bear in mind that the assurance factor can change with time (for example, due to the structure wear), and now the task is set of ensuring certain preset assurance factor value during the schedule service period with no relation to the pattern of assurance factor change with time (Fig.1.6.).

Under these conditions the structures with the actual value of the assurance factor

$$k(t) = P(t) / [P]$$

defined by the plot 1 or plot 2 (Fig. 1.6), the both realizing the required value $k(t) > [k]$ in all the time interval, though in case 1 the structure is more reliable in general. This fact is not allowed for at all in the current standards.

The values of partial reliability indices fixed in codes, in whatever way they have been established, are oriented to a certain “averaged object” of design. But the consequences of the failure of one and the same structure, used on the objects for different purposes, may differ widely that is to be taken into account in the design.
The fail-safety level differentiation by the structure reliability criterion is performed through a relatively new coefficient of the method of the limit-state analysis – reliability factor $\gamma_n$ (designated reliability factor, significance index).

The home specification base uses classification of building projects according to reliability level (DBN B.1.2-14-2009), and a coefficient $\gamma_n$ introduced in the system of design coefficients of the method of limit-state analysis by the Soviet building code in 1981. It is the important element of design reliability control and differentiation of reliability level as to the estimate of the design object significance. In Russia it is fixed in the Law *Technical Regulation of Safety of Buildings and Structures* which includes the following requirement:

*The accounts justifying safety of the adopted constructive decisions for a building or structure should be made with allowance for the level of reliability of the designed building or structure. With this in mind the design values of forces in the members of building structures and the building or structure foundation should be determined with allowance for the designated reliability factors*....

Later on similar methods of reliability differentiation appeared in other foreign specifications. The Eurocode system uses the factor $K_{F1}$ which coincides in content with $\gamma_n$. American standards ASCE-7 use a system of Importance Factors.

**1.6. Use of reliability theory**

First works on using statistical methods in structural mechanics were directed to substantiation of values of the safety factor – a
single, in the stress method, or differentiated, as in the method of limit-state analysis - but they did not concern the approach to providing the structure serviceability.

For example, M.F. Khotsialov, who worked at the construction site of Svir hydroelectric station, took into consideration the spread of block strength of concrete placed into the dam, and considering load as the determined one, he defined a required safety factor that guaranteed, with some set probability, the structure indestructibility [Khotsialov, 1929]. M.S. Striletsky [Sreletsy, 1947] used in his works not only strength parameters but also load ones as random values. The rest was based on a comparison of maximum possible system reaction with its minimum possible carrying capacity, and a not too distinctly determined value, the so-called “indestructibility guarantee”, was used as the comparison criterion. Such an approach was used under development of specifications for the method of limit-state analysis. The general shortcoming was that the factors at the value of loading factor (overloading) and at the values of supports (homogeneity) were determined for each design factor independent of variability of other factors.

They had rejected this proposition; the idea of probabilistic analysis of the limiting state was advanced. Two components of this approach were discussed under its formation and development:

• choice of the limiting state model;
• legitimacy of using the probabilistic analysis.

As to the model, orienting (first of all) to the condition of ensuring the strength in 1947 O.R. Rzhanitsin [Rzhanitsin, 1947] and A.M. Freudental [Freudental, 1947] submitted for consideration (practically simultaneously) the indestructibility function in a form of the carrying capacity factor

\[ \tilde{S} = \tilde{R} - \tilde{Q}, \]

which, by force of occasional character of the load parameter \( \tilde{Q} \) and resistance parameter \( \tilde{R} \), is also a random value. Considering that \( \tilde{Q} \)
and $\tilde{R}$ have a normal Gaussian distribution with mean values $\bar{Q}, \bar{R}$ and dispersions $\tilde{Q}, \tilde{R}$, they obtained the expression for probability of fulfilling indestructibility inequality

$$prob\left( \tilde{R} \geq \tilde{Q} \right) = prob\left( \tilde{S} \geq 0 \right) = 1 - \Phi\left( \frac{1 - k}{\sqrt{v_R^2 + k^2 v_Q^2}} \right),$$

where $v_R^2 = \bar{R}/\bar{R}^2$, $v_Q^2 = \bar{Q}/\bar{Q}^2$ – squares of variability coefficients, $k = \bar{R}/\bar{Q}$ – probability safety factor, $\Phi(\cdot)$ – Laplace function.

Certain values of failure probability being set

$$V = prob\left( \tilde{R} \leq \tilde{Q} \right),$$

we can obtain all required factors of the method of limit-state analysis. In particular, using A.R. Rzhanitsin’s indestructibility guarantee value $\gamma$, equal to inverse value of the safety factor

$$\gamma = \frac{\bar{R} - \bar{Q}}{\sqrt{\bar{R} + \bar{Q}}} = \frac{k - 1}{\sqrt{v_Q^2 + k^2 v_R^2}},$$

the safety factor may be expresses through this value

$$k = \frac{1 + \sqrt{\gamma^2 v_Q^2 + \gamma^2 v_R^2 - \gamma^4 v_Q^2 v_R^2}}{1 - \gamma^2 v_R^2}.$$

Then the material safety factor and load factor will be equal to

$$\gamma_r = 1 - \gamma v_R, \quad \gamma_q = 1 + \gamma v_Q.$$

It should be noted that the use of the failure model connected with using the indestructibility function $\tilde{s}$, is not absolutely necessary. Numerous researchers considered other models. Thus, for example R. Levi [Levi, 1949] used the demand to safety factor in direct form $\tilde{k} = \bar{R}/\tilde{Q} \geq 1$, B.I. Snarskis [Snarskis, 1962,
1963] used it in inverse form \( \tilde{k}^{-1} = \tilde{Q}/\tilde{K} \leq 1 \) and K.A. Cornell [Cornell, 1969], A.Ya Driving [Driving, 1973] – in logarithmic form \( \ln(\tilde{k}) = \ln(\tilde{Q}/\tilde{K}) \geq 0 \).

This approach by O.R. Rzhanitsin and A Freudenthal was later called in western literature the second-moment method and was assumed as the basis in design of structures with preset reliability. The work by C.A. Cornell [Cornell, 1969], where this method was presented in general form and where a simple reliability model with two random values \( \tilde{Q} \) and \( \tilde{K} \) was generalized for the case of \( n \) values, played a significant part.

Cornell introduced a linear function of the limit state of \( n \) independent random variables (geometrical, topological and mechanical parameters of the element and load parameters)

\[
S(\tilde{X}) = a_0 + a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + \ldots + a_n \tilde{x}_n,
\]

(an analogue of carrying capacity reserve in terminology of O.R. Rzhanitsin). It divides the parameter space into two zones – safety zone, where \( S(\tilde{X}) \geq 0 \), and a zone of failure, where \( S(\tilde{X}) < 0 \). In \( \tilde{x} \) coordinates the indestructibility guarantee, which Cornell had called the safety characteristic \( \beta \), is the shortest distance from the coordinate origin to the plane \( S(\tilde{X}) = 0 \).


All the above models are connected with the notion about the limiting state, as the reaching of the ultimate strength, and, though a list of situations connected with limiting
states is much longer, it was desided to agree with such approach as
the first approximation for substantiating standard requirements.
The probabilistic approach itselfs had caused serious discussions.
When proceeding from the frequency model
of probability, there arise doubts as to this
notion applicability to individual order events,
com since each significant structure is unique.

This problem was discussed even in the
late 80’s of the 20th century. In particular,
some experts, which expressed certain
sccepis, said “... a customer will fail to
understand information that 10% of the total
number of shafts could fail. And he will demand a spare shaft just
for his car”. (Strelnikov, 2000).

Bajeesian interpretation of probability as a certain objective extent
of confidence in the judgement truth which holds its sense, no matter
whether the situation is analyzed in mass, put an end to such doubts.

A question of insufficient consistency of the used statistical data,
especially in cases of the so-called distribution tails, arose as another
problem. In this connection a considerable attention was drawn to
improbability methods of estimating indeterminateness; the most
significant works of this trend belong to I. Ben-Haim and I.
Elishakoff [Ben-Haim & Elishakoff, 1990], [Ben-Haim, 1994]. As a
rule, these estimates are of the range nature, they do not object and
do not underestimate the probabilistic determination of reliability, but
they proceed from the assumption that probability is not the only
fiducial point in estimating reliability. As regards this Ben-Haim
[1990] said that the probability system is reliable, if the probability of
admissible behavior is rather low. In improbabilistic formulation
the system is reliable, if the range of its behavior fluctuations is
admissible.

A new stage in using the reliability theory was marked by the
works by V.V. Bolotin [Bolotin, 1965, 1971], where time was
included in the problem. Considering parameters, which determine

Masanobu Shinozuka

![Image of Masanobu Shinozuka]
both the external effects and the structure resistance as random processes, and using inequality

\[ \tilde{Q}(t) \leq \tilde{R}(t), \quad 0 \leq t \leq T, \]

V.V. Bolotin thinks that the failure is the intersection of the level \( \tilde{R}(t) \) by the process \( \tilde{Q}(t) \). He formulates the problem of reliability as the estimation of failure probability at the set time interval \([0…T]\); for this purpose he estimates the expected number of such intersections in a time unit. Later on M. Shinozuka [Shinozuka, 1964] presented V. V. Bolotin’s theory in more extended form.

Bolotin obtained in his work [Bolotin, 1977] the safety factor value \( k = \frac{\tilde{R}}{\tilde{Q}} \) for the structure, which strength is a random value \( \tilde{R} \), while a load \( \tilde{Q}(t) \), affecting it is a normal stationary random process with mathematical expectation \( \bar{Q} \), dispersion \( \sigma^2 \) and efficient frequency \( \omega : \)

\[
k = 1 + v_Q \sqrt{\frac{2(1 + \tilde{R}/\bar{Q}) \ln \left( \frac{\omega T}{2\pi V} \cdot \frac{1}{1 + \tilde{R}/\bar{Q}} \right)^{1/2}}},
\]

Here \( T \) – established operation time, and reliability with nondestruction probability \( V \) should be provided for it.

It should be noted that all the above approaches with using the notion of carrying capacity margin practically consider the fail safety of a certain “intersection”, the load parameter \( \tilde{Q}(t) \) and resistance parameter \( \tilde{R}(t) \) being determined for it. Under these conditions some general system variants of failure, for example, something like a loss of general stability, are not considered.

Use of the methods of reliability theory in the structure design [Bolotin, 1971] resulted in
posing new problems connected with analysis of the effect of the system member reliability on its general reliability.

This problem was posed for the first time in 1943-1944, when Germans analyzed failures of FAU rockets – one of the most complex systems, which existed at that time.

It was noted that, though certain units of the rocket are rather reliable, only a quarter of FAU launches were successful. German mathematician Eric Pieruschka, who worked in the laboratory of Wernher von Braun, proved that the rocket reliability rather equals the product of reliabilities of all its components, than reliability of the most unreliable element, as Braun thought. There arose the notion of parallel connection and redundancy of elements.

The variants of series and parallel connections of the system members are considered in classical theory. But in complex building structures the connections of their members may be classified as parallel or series ones in rare cases. The failure of one or even several elements results in the change of the design model, but the latter continues working. An analysis of the corresponding reliability problem is much more complex than for the systems under consideration of classical reliability theory; that is why the elementwise reliability verification is still used in the design practice, that is the conception of a “weak link”, which failure is considered as crucial. Such approach has been fixed in all design specifications.

At the same time survivability [Perelmutter, 2007] of reliable structures is analyzed in recent years, when considering some extreme (beyond project) conditions of their existence; and that has become the standard demand in the last decade. This analysis is performed with allowance for the out-of-limit states of certain elements, and its aim is to prevent the so-called “nonproportional fracture”, when a local failure can result in the global collapse.

Probable errors, connected with the lack of adequate data in reliability calculation, have been discussed in the works [Ben-Haim, Elishakoff, 1990], [Elishakoff, Hasofer, 1996]. But this important
subject has not drawn attention of experts in the field of stochastic mechanics.

1.7. Models of loadings

Over a long period of time probabilistic studies of reliability operated on random values of loads and supports, but only one load effect was considered in practice. Its values at some instants of time were considered to be certain realizations of a random value. Their set gave the values of such statistical parameters as mean value, root-mean-square deviation, etc. Proceeding from these characteristics the design load values were obtained, which provided for a certain preset probability of their appearance. If more than one such load was taken into account, the hypothesis was admitted of simultaneous attaining maximum (design value) for different loads.

Consideration of loads in the form of random time functions (random processes) is more realistic.

First studies, where loads were presented as a certain random time function, naturally belonged to the effects, which quickly changed in time, such as seismic load and the effect of rough sea. J. Housner [Housner, 1947] described the soil acceleration during the earthquake as a sequence of uncorrelated pulses. Presentation of load and stresses in the structure as continuous random processes was probably used for the first time in ship building [Yekimov, 1957].

Presentation of loads as random sequences of different kind proved rather promising. Probability model of N.S. Striletsky in a form of a random series of loads variability [Streletsky, 1947], outlined in 1947 and formulated in detail in 1966, was one of such first presentations. Such a series may be natural, corresponding to actual sequence of loads, or sorted out; force actions in this series are plotted in the order of growth.

The history of load change \( Q(t) \) is presented by the sequence of rectangular pulses with constant duration \( \tau \) and random values of ordinates \( \tilde{Q} \) in the model of Ferry Borges-Castanheta [Ferry Borges,
Castanheta, 1968] offered in 1968. These ordinates are chosen to embrace the actual course of loading process (Fig. 1.7, a). If the correlation function is damped over the chosen time interval $\tau$, the sequence of random values may be considered statistically independent one.

O.R. Rzhanitsin had extensively studied a more general three-parameter model called a discrete presentation of loads, though he did not define the method of assignment of the values $\hat{Q}$, which he called overload [Rzhanitsin, 1978].

One can disregard the change of loads in time and consider the design load values as random values. A series of works by A.R. Rzhanitsin published in 1949 was one of the first studies dedicated to connections of thus presented random loads.

He had practically solved the problem of the total action of several independent loads, each of them being a random value with Gaussian distribution. As a result, the following expression was obtained for the combination coefficient:

$$\psi = \sum_{i=1}^{n} \frac{C_i}{1 + \gamma_i V_i} + \gamma_n \sqrt{\sum_{i=1}^{n} \left( \frac{C_i V_i}{1 + \gamma_i V_i} \right)^2 + 2 \sum_{i \neq j} \frac{C_i C_j V_i V_j r_{ij}}{(1 + \gamma_i V_i)(1 + \gamma_j V_j)}},$$

where $V_i$ – variation coefficient of $i^{th}$ load; $\gamma_i$ and $\gamma_n$ – standardized deviations of the $i^{th}$, $n^{th}$ and total design loads; $C_i$ – the number of effect (part) of $i^{th}$ load; $r_{ij}$ – correlation coefficient of $i^{th}$ and $j^{th}$ loads.

Fig. 1.7. Presentation of load by the sequence of values
In case of independence of combined loads the second summand under the radical disappears.

This coefficient values presented in the Building Code with allowance for O.R. Rzhanitsin’s propositions were determined as follows:

\[
\psi = \left[ \sum_{i=1}^{n} q_i^n + \sqrt{\sum_{i=1}^{n} \left( q_i^n \right)^2 \left( \gamma_{fi} - 1 \right)^2} \right] / \sum_{i=1}^{n} q_i^n \gamma_{fi},
\]

where \( q_i^n \), \( \gamma_{fi} \) – regulatory value and reliability (overload) index of \( i^{th} \) load.

The case of allowance for load variability in time was first strictly and evidently studied by V.V. Bolotin. In 1962 he considered the problem of summing up the effects of several, properly speaking, correlated, random Gaussian processes [Bolotin, 1962]. His considerations were based on the analysis of probability of the total random process surpassing a certain level which should be independent of the number of accounted components of the total process. In the same manner as for separate loads, the frequency composition of the random effect, caused by the effect of combined loads presented in the form of stationary random processes, is determined by the effective frequency.


A strict approach to solving the probabilistic problem on combination of different loads and on statistical properties of the effect of their total action requires a rather keen and detailed analysis of properties of those random processes, which describe behavior of each load. And the authors [Gnedenko et al., 1969] had probably reason to state that a chief thing is “... a search of comparatively simple solutions for the adopted models in a form of random...
processes and presentation of these solutions, if possible, in a form of operations over random values”.

Such was the motivation of the work by Turkstra [Turkstra, 1970], where the author attempted to substantiate the corresponding regulations of combining the random loads, which change with time. His considerations were based on the Ferry Borges-Castanheta’s model, the history of the load $Q(t)$ change being presented in it by the sequence of rectangular pulses.

The Turkstra rule is confirmed by the experience and observation, since the failure cases mostly appear, when one of the loads reaches extreme value, and they appear too rarely under the effect of a combination of extremes of several different loads that change with time.

The great number of works were dedicated to investigation of possible combinations for the certain kind loads. Among them one can distinguish some great classes of loads, for example, such as climate effects on buildings and structures, or ridge crane loads in industrial buildings. Such research was usually made with the aim to find certain combination coefficients which were to be included in design codes.

1.8. Optimal reliability level

The use of probabilistic methods in any form calls for a distinct concept of admissible level of risk. The decision making concerning this level is one of the basic problems. The evolutionary trend connected with the “trial-and-error” methodology and with analysis of accumulated experience is still the most operative one.

It is relevant to cite the statement of well-known experts in the reliability theory B.V. Gnedenko, B.A. Kozlov and I.A. Ushakov; they note: “… when determining design parameters of carrying structures simultaneously with using the methods of reliability theory ... the decision is made as based on experts’ intuition, confirmed by the analysis of the existing level of product quality characteristics. Of
course, some errors frequently occur, but there is certain “natural selection” in the general process of engineering development; as a result the wrongly designed products “die off”. The appropriate norms of numerous characteristics, including reliability ones, are formulated in such a way” [Gnedenko et al., 1969].

But opinions are heard in increasing frequency in favor of optimization solution of the above problem.

Karl Forssell was the first to write about that even in 1924 [Forssell, 1924]. He had unambiguously (though with caution) formulated the concept of economic determination of safety levels and a choice of a certain reasonable risk in building. But the then methods of solving optimization problems did not allow not only solving but even formulating accurately the problem.

When it comes to economic losses as a result of failure (the so-called systems with purely economic reliability), everything may be settled by comparing expenditures for providing the required level of reliability and a possible harm from possible unreliability of the object. Such propositions were made by many authors (see, for example [Rzhanitsin, 1973], [Hilton & Figen, 1960], [Moses & Kinser, 1967], [Switzky, 1964]); in so doing they used various admissions which concerned functional dependence of a possible loss on the safety factor. They most frequently used the criterion of minimum expenditures for initial erection of the structure plus mathematical expectation of loss from possible faults and failures in the course of the building service.

Complete expected expenditures increase much more quickly under the increase of failure probability (compared with optimal value) than under the decrease of this probability, i.e. the reliability excess is less expensive than its deficit. That is the important peculiarity of this target function. Maybe engineers, when taking (in case of doubt) decisions on ”the safety margin” are guided by these problem peculiarities, which were apprehended at a half-intuitive level.
The research of A.Ya. Driving [Driving, 1973] was a typical work of this trend brought (in contrast to many others) to final result and introduced in some normative documents. The author obtained optimal value of the safety factor $k = \frac{R}{\bar{Q}}$ on condition that the total loss from the failure is expressed as a formula

$$ U = C_0 k^{1-\theta} + u. $$

Here $C_0$ – the structure cost at $k=1$, $0 \leq \theta < 1$ – coefficient which depends on the stress state kind ($\theta = 1/3$ under bending), $u$ – «side loss», caused by the structure failure and practically independent of $k$. Under the assumption on logarithmically normal distribution $\tilde{Q}$ and $\bar{k}$ and on the fact that mathematical expectation of undestructibility function $\tilde{S} = \bar{R} - \tilde{Q}$ is close to one (then $v_S \approx \bar{R} + \tilde{Q}$), for great values of the coefficient of economic reliability $\xi = u / C_0$ optimum value of $k$ is determined as a formula:

$$ \ln\left(k_{opt}\right) = \sigma \sqrt{\frac{\sigma^2 (1-\theta) (2-\theta)}{1-\theta}} \frac{\frac{\xi T}{\sqrt{2\pi}}} + \frac{\xi^2 (3-2\theta)}{2}, $$

where $T$ – service life, while parameter $\sigma$ is subject to the ratio $\sigma^2 = \ln(1 + v_S)$.

If possible failures or faults of the whole structures and their members are the cause of human victims or traumas, as well as of inestimable losses of the art, historical and other values, psychological consequences of possible accidents should be also taken into account. There arises a problem of determining the “value of life”.

Bruno Snarskis was one of the first in 1962, who concerned this problem. He considered a certain hypothetic way of such problem solution. Snarskis proceeded from the idea that no society (in the past, present or future) possesses limitless resources; it can spend only a certain sum for health and life of its members that may be estimated by analyzing the country budget. Thus we can estimate the “value of life”.

49
But experts often raised objections as to probable posing the optimization problem which concerned uneconomic reliability.

That is the problem statement in the well-known monograph by G. Augusti, A Baratti and F Cachiati: “...not all kinds of losses can be estimated in terms of money. Nevertheless many researchers try to solve the problem with the help of the target function, obtaining absurd results. The attempts to include the human life value into the target function as an additive to total expenditures bring to the same results” [Augusti et al., 1988, p. 309].

The authors notice that optimization following the criterion of minimization of total expenditures \( H_{\text{tot}} \), the value of life being included in them, results in the condition \( \Delta H_{\text{tot}} / \Delta v = 0 \), where \( \Delta v \) – the number of saved lives, the expenditures being increased by \( \Delta H_{\text{tot}} \).

It is stated that “this condition may be interpreted as follows: the community has no interest in paying just a dollar for additionally saved life” and it is concluded that the optimization solution is practically unacceptable.

But the above condition may be formulated in another way that is not so cynical. For example: “the community has no interest in paying just a dollar for some life saved in the considered way, since using limited means in other way (e.g. organizing early cancer diagnostics), it is possible to save more lives”.

Maybe such formulation removes ideological component of the discussion, but it does not solve the life value problem, which numerical solution is taken arbitrarily and is in no way argued.

It seems that the approach to solving the problem on comparison of economic and social expenditures which is based on the following arguments proposed in the work [Black and Niehaus, 1980] is one of most successful ones. The authors have noticed that expenditures for safety are connected with some actions (construction work,
installation of protective equipment) fraught with possible loss of human health and life that is confirmed by statistics of accidents in national economy. Based on the analysis of information in Germany S.C. Black and F. Niehaus came to a conclusion that $12.28 \times 10^{-3}$ deaths fall to each 1 million dollars spent for equipment production and works for safety, as a result of:

Accidents in industry – $7.86 \times 10^{-3}$;
Accident crash – $4.12 \times 10^{-3}$;
Occupational diseases – $0.306 \times 10^{-3}$.

Thus the problem is reduced to a single dimension by the system “life for life”. It was found out that the expenditures over 33 million dollars for one saved life really result in the risk growth. This proves to be “too safe” solution.

An alternative approach was proposed by O.R. Rzhanitsin [Rzhanitsin, 1973] and developed in the works by V.D. Raizer [Raizer, 2008]. They stated that, if structure reliability is optimized under uneconomic loss, one can calculate “the number of saved lives” for the structure with such reliability and thus obtain economic equivalent of uneconomic losses, corresponding to the obtained reliability value. The already available structures may be subject to inverse estimate; then the conventional economic equivalent may be determined, which (maybe not consciously and intuitively) was introduced by builders at different times for structures of different reliability.

### 1.9. Failure criteria

In accordance with ideology of structure design by the method of limit states it is foreseen that the building or structure service life is stopped prior to exhaustion of its actual carrying capacity, and this fact is declared in one of postulates of the method of limit state analysis. The criteria which determine the limit state were formulated in different ways and in different years.
For example, Baldin [Baldin et al., 1951] wrote in his pioneer work that the limit state “... is state of the structure, when its further service is impossible”. This definition concerned all the limit states, but the first limit state was associated with its carrying capacity (structure strength and stability, material fatigue).

Later on, for example “… the limit states which lead to complete nonserviceability of structures, basements (of buildings or whole structures) or to complete (partial) loss of carrying capacity of buildings and whole structures” are referred to limit states of the 1st group by GOST 27751-88, which has been in force for many years. This formulation foresees that, despite the loss of carrying capacity, such events as service ceasing because of economic inexpediency of further maintenance may also take place as the 1st group limit-state criterion. Such was the end of service of Kyiv TV centre tower built in Khreshchatyk Street in the early 50’s, since the transition to new type antennas was connected with too expensive modernization. In other words, the formulation of GOST 27751-88 implies both physical and moral depreciation as a cause of transformation to the 1st group limit state.

Note, that the term “complete object inserviceability” in each specific case demands some definition. If water supply goes out of service in some old building it should be decided whether it is worth to install new pipes or to demolish the building. If the building is of historical interest, it may be turned into untenantible museum premise after establishing the its uninhabitability.

A lot of well-known old buildings and memorials have actually gone out of service. For example, the pyramids of Giza are often mentioned as an example of long-living ones, but they do not and cannot serve their purpose, since the surface of their faces is failed (Fig.1.8).

It formally appears that accidental failures are, to some extent, non-normative events, and statistics of accidents cannot be used for estimation the actual level of reliability, since it does not include numerous cases of failure-free removal of structures from service. The
compilers of International Standard [ISO ST 2394] were more consistent; they have determined that “3.1.1… the 1st group limit states which correspond to maximum carrying capacity (related to safety)”, or of Eurocodes [EN 1990; 2001], where it is indicated that “3.2. (1) Ultimate limit states are the limit states associated with break down or other similar kinds of failure. (2) The states just before the destruction and, for the sake of simplicity, considered instead of break down as such, also belong to the ultimate limit states. (3) The ultimate limit states refer to: safety of the structure and its surrounding; safety of people”

Fig. 1.8. Pyramid of Hefren in Giza (the 27th century B.C.) – present appearance

A new Russian standard [GOST P 54257] has adopted analogous attitude, the following limit states are introduced:

- the first group of limit states – the states of building objects, which exceeding leads to the loss of carrying capacity of building structures;

- the second group of limit states – the states, which exceeding disturbs normal service of building structures, exhausts their life resource or disturbs comfortness;

- special limit states – the states which appear under particular effects and situations, and the exceeding of these states results in the failure of buildings and structures with catastrophic after-effects.
If the first limit state is associated with attaining a critical value by some structure element (attaining the ultimate strength, loss of balance stability, etc.), some indeterminacy is observed as to the second limit state, and setting any limits is something conventional. But in the recent years (see, e.g. [EN 1990: 2001] the attempts are made to differ reversible and irreversible variants of the second limit state.

Irreversible limit states (Fig. 1.9, a) are such that do not disappear after ceasing the effects, which have caused them (for example, local damage or concrete creep flow). And reversible limit states (Fig. 1.9, b) do not develop, and sometimes disappear, when the effects, which have caused them, are stopped (for example, displacement under the effect of wind loading or excessive vibration).

There is a part of time established for reversible limit states, during which this state may be disturbed. For example, this occurs for sharply directed radio communication, when the observable angles of antenna rotation lower the reception quality; then it may be conventionally taken that such deterioration of communication quality is admissible for 5% of time. Such an approach was first offered in the work [Savitsky, et al., 1968] and regulated by Ukrainian building specifications [DBN B.1.2-14].

![Deformation vs. Time](image)

**Fig. 1.9. Variants of the second limit state**

Limit values are established for irreversible limit states of the second group from the viewpoint of serviceability; in some cases
they can be determined rather objectively, but in most cases such limits are conventional in a certain sense. It is difficult, e.g. to prove that a sag of 1/250 of a span is admissible, but at the value of 1/245 it is already inadmissible.

Service normalcy (that is the characteristic of the 2nd group limit states) rarely becomes complicated immediately after overcoming the established limit. Most often the limit determining transition to such a state is fuzzy and the proper failures are indistinct.

But a probable complication is most often connected with the “depth” of penetration to out-of-limit area, since in situations with fuzzy failure the controlled boundaries are conventional, and there is a monotonous dependence of the loss on the extent of prohibition violation.

To consider this property a model of fuzzy failure was proposed in the work [Holicky & Ostlund, 1996]. According to this model the extent of meeting the requirement of service quality $V_x$ linearly decreases overcoming the threshold value $r_1$.

It is considered that the actual value of quality index of “normal service complication” is a random value with average $(1 - V_x)$ that is shown on Fig. 1.10. b.

![Diagram](image)

Fig. 1.10. On estimate of the fuzzy failure loss

The quality loss index is introduced
\[ D_r(x) = \frac{1}{N} V_r \left( \int_0^x \varphi(z|V_r) \, dz \right) \, dV, \]

where \( N \) – normalizing factor, which helps represent the functional \( D_r \) on the interval \((0,1]\). Rather representative comparisons of estimated values of \( D_r \) with a number of known designs of buildings and structures, which, in the experts’ opinions, are comfortable and convenient, could allow putting the used scale (e.g. for admissible saggings) in line with \( D_r \) function scale values. Such a comparison was done in the above mentioned work by M. Khilitsky and L. Ostlund, where saggings of interfloor ceilings were considered as an example.

1.10. Conventional and real safety factor

It has already been noted that the method of limit states is based on the idea of refusal from detailed analysis of all ordinary (normal, serviceable) states of structures. Concentration on failure states with putting emphasis on the first limit state as such determining the structural form is not the only method advantage.

Since main lifetime of a structure corresponds to the state of normal service, and destructive changes in the structure materials occur just for these states (for example, corrosion processes or accumulation of fatigue damages), the analysis of the structure, being in normal operation and far from the exhaustion of strength and stability, becomes dominating in the context of providing structure serviceability and longevity. The analysis in service period may be determining for most structure parameters. On this point Iosilevsky [Iosilevsky, 1999] wrote: “A loss of designed control over the structure in the period of transition from “healthy” (normal, serviceable) state to the limiting state is nothing but a crash in methodology of designed prediction of behavior of carrying
structure under load... Logic vacuum formed between the operation and limiting (accident) state is inadmissible”.

One can suppose that availability of checks as to the second limiting state removes this methodological crash, but the point is that this group of states is also a limiting one, i.e. it corresponds to rather rare ultimate states of parameters of the structure and environment. For example, the regulatory values for the structures operating under the effect of snow or wind load are realized once every five-six years and are too far from the normal service condition.

In most cases the basic inequality of the method of design limiting states is presented in the form

\[
\psi \gamma_n \gamma_f F_n \leq \gamma_c R_n / \gamma_m
\]

where \( \psi, \gamma, \gamma_f, \gamma_m, \gamma_n \) – load combination coefficient, operation conditions coefficient, reliability index under load, material reliability index, and structure reliability index, respectively; \( F_n, R_n \) – normative value of generalized effect and of generalized resistance, the limiting states being estimated by them.

Some researchers (see, for example, [Baldin et al., 1951], [Gvozdev, 1964]) identify the value

\[
K = \psi \gamma_n \gamma_f \gamma_m / \gamma_c
\]

with standardized safety factors of the system.

It is not difficult to note that the coefficient \( K \) is not much different from unity (it fluctuates most often within 1.15-1.25, though for some kinds of the structure load and material the upper limit of this interval may even be higher), that provides for the coincidence of the design limiting state with true serviceability limit of the structure, though it is not so in reality. True safety factors far exceed the unity, since actual serviceability limit differs from conventional one that appears as the design limiting state. The value \( K \) does not meet a true margin of carrying capacity mainly because its real exhaustion is usually connected with a series of nonlinear effects, which
redistribute forces in the system, when it approaches the failure. As a result of such redistribution the estimate obtained using $K$ and calculated using another design model (linear, as a rule), may be both overrated and underrated.

Thus we see that the use of the method of design limiting states does not define the system behavior at the service stage (that was mentioned above) and estimates rather approximately that margin, which isolates the transition from the design limiting state (often conventional) state to the true limit of the system carrying capacity.

To draw more consistent inferences we should make experiments and/or special analysis performed by the methods different from those given in normative documents. So, it should be stated that the design limiting state is separated from the real failure state by a certain barrier and its value is most often unknown.

1.11. The design calculation cannot allow for everything

We know numerous cases of considerable exceeding of design value of a certain load which did not result in the accident failure. There arises a natural question as to causes of this phenomenon.

Most often it is because any structure is rarely designed reckoning on the action of just one load. In so doing the load factor $\gamma_f$ does not allow for all factors determining a real pattern of load change with time. Thus, the building code gives the value for the crane load $\gamma_f = 1.2$ that is too far from reality. Standard values of $\gamma_f$ are determined for “loads in general” and do not allow for a whole number of additional random parameters, which arise under analysis of the method of such load implementation in a certain structure.

Such a factor as probability of fully concrete location on the influence line, when the design force may be realized in the structure, may be important for the crane loading. The joint action of several cranes being accounted, it appears that the probability of coincidence
of their design locations is several times less that sharply decreases the mean level of force in the structure.

The above peculiarities of real loading of structures are only slightly allowed for by the values of load combination coefficients which are also calculated for “loads in general”. Additional reserves of carrying capacity are most often created under these conditions.

It may be supposed that such additional reserves allow ignoring some normative constraints. But the fact of violation of limit inequality indicates that the structure operates in conditions, which were not foreseen by the designer, i.e. the latter had not calculated and analyzed them. The safe service is not assured in these conditions. That is undesired, even if there was no accident, especially allowing for that the above reserve value is rather the evaluating fact than the assured one. Most structures have no such reserves, especially now, when production and mounting quality and thoroughness of material property control are too low.

It should be remembered that, besides the sharply defined loads and effects, a certain load or influence may occur provided neither by normative documents nor by designer’s prediction as to occasional influence on the designed structure. In this context the notion “influence” is rather comprehensive and includes, for example, such cases as coarse defect, personnel error, natural phenomenon uncommon for the given district, in word, various surprises having grave consequences. Undoubtedly, these effects are not mass ones, and thus, their statistical analysis as well as the account of their influence in probabilistic analysis of reliability are difficult. A probable trustworthy hypothesis of probability type is that their realization is equiprobable in the course of time and they are rather rare phenomena. These effects are so rare that do not usually get into the sampling, which is a basis for determining statistical parameters (Fig. 1.11,a), or they are not of the probability nature. American economist Nicolas Nassim Taleb has called these events “black
swans”3. From the viewpoint of these “surprising” events the structure susceptibility is its important characteristic.

![Diagram](image)

Fig. 1.11. On probabilistic estimate of very rare events

Susceptibility is the parameter which characterizes a possibility of inflicting damages of any nature on a given structure in different ways or with different factors. Susceptibility is inseparably connected with the well-known characteristic “survivability” and with “mobilizability” – a characteristic proposed additionally in the work [Perelmutter, Pichugin, 2014].

As for the building projects the notion of survivability began developing much later than in other fields of engineering. This is connected with comprehension of the fact that some unreckoned accidental effects may always can cause local failures. This paradigm was primarily realized in respect of earth-quake-resistant construction (though the term “survivability was not necessary here), and in particular, there appeared an idea of distinguishing the so-called main carrying structures: their reliability saves a building or structure from the total fracture under accidental influences, even if its farther designated use proves impossible without major repair.

---

3 Juvenal said “raraavis in terries nigroque simillima cygno (Lat.)- “a good man is as rare as a black swan”, since there was a hypothesis that all swans are white. It was considered true until a black Australian swan was found in 1700.
The structure survivability may be considered assured, if the first local failure results in the fracture of a limited site, which admissible size is regulated by specifications or has been co-ordinated with the customer.

The explicit requirement of ensuring survivability was first proposed in 1998, when developing the Building Code [DBN B.1.2-14] draft, but it was done only in 2009 after long discussions and bureaucratic procrastinations.

If it is usually thought that survivability is a somewhat spatial characteristic, which could show how local perturbations were distributed in the system space and whether the local failure could develop disproportionately in breadth, the mobilizability is considered as a temporal characteristic, which shows how much the system is ready in its service time and is able to respond to the local in time (pulse) sudden perturbation.

A noticeable lack of the structure mobilizability as well as insufficient survivability have to attract intent attention of designers and to make them to take certain precautions.

References


Bernshtein, S.A. (1957), *Ocherki po istorii stroitelnoy mekhaniki* [Essays in the history of structural mechanics], Gosstroyizdat, Moscow, Russia.


Bolotin, V.V. (1965), Statisticheskie metody v stroitelnoy mekhanike [Statistical methods in structural mechanics], Stroyizdat, Moscow, Russia.

Bolotin, V.V. (1972), Primenenie metodov teorii veroyatnosti I teorii nadyoznosti v raschotakh soorugheniy [Use of the methods of probability theory and reliability theory in design of structures], Stroyizdat, Moscow, Russia.


Vitruvy Mark Pollion (1936), Desyat knig ob arkhitekture [Ten books on architecture], translated by F.A. Petrovsky, Vol. 1, Izdatelstvo Vsesoyuznoy Akademii arkhitektury, Moscow, Russia.

Galilei, G. (1964), Besedy i matematicheskie dokazatelstva kasayushchiesya dvukh novykh otrasei nauki. Izbrannyie trudy v dvukh tomakh [Discussions and mathematical proofs concerning two new fields of science. Selected works in two volumes], Vol 2, Nauka, Moscow, Russia.


Iosilevsky, L.I. (1999), Prakticheskie metody upravleniya nadyozhnosti zhelezobetonnykh mostov [Practical methods of reinforced concrete bridge reliability control], NIIs “Inzhener”, Moscow, Russia.


Loleit, A.F. (1933), Rezultaty opytnoi proverki osnovnykh poizdaniy raschota izgibayarmykh zhelezobetonnykh elementov no printsipu kriteskikh usilii [Results of experimental check of basic propositions of design of bending reinforced concrete elements on the principle of critical forces], Litografiya, 38 p.

Perelmutter, A.V. (2007), Izbrannye problemy nadyozhnosti i bezopasnosti stroitelnykh konstruktsiy [Selected problems of reliability and safety of building structures], Izdatelstvo Assotsiatsii stroitelnykh vuzov, Moscow, Russia.

Perelmutter, A.V. (2011), Ocherki po istorii metallicheskih konstruktsiy [Essays in the history of metal structures], Izdatelstvo Assotsiatsii stroitelnykh vuzov, Moscow, Russia.


Streletsy, N.S. (1936), “Once more on the problem of structure safety factor analysis” Proekt i standart, no. 3.

Streletsy, N.S. (1936), Novaya metodika raschota konstruktsiy [New methods of structure design], Izdanie Moskovskogo inzhenerno-stroitelnogo instituta, Moskow, Russia.

Streletsy, N.S. (1947), Osnovy statisticheskogo uchota koeffitsienta zapasa prochnosti sooruzheniy [Principles of statistical account of safety factor of structures], Stroyizdat, Moskovia, Russia.

Streletsy, N.S. (1966), K voprosu razvitiiya metodiki raschota po predelnym sostoyaniyam [On the problem of development of limit states design methods], MISI, Moskow, Russia.


U 28-42/Narkomstroy (1942), Ukazaniya po proektirovaniyu i primeneniyu stalnykh konstruktsiy v usloviakh voyennogo vremeni [Directions on design and use of steel structure in war time conditions], Stroyizdat, Moskow, Russia.

U 37-42/Narkomstroy (1942), Ukazaniya po proektirovaniyu i primeneniyu betonnykh i zhelezobetonnykh konstruktsiy v usloviakh voyennogo vremeni
[Directions on design and use of concrete and reinforced concrete structures in war time conditions], Stroyizdat, Moscow, Russia.


Coulomb, C.A. (1773), Application des regies de maximis et minimis a quelques problemes de statique, relatifs a l'architecture, Memoire des savants etrangers de l'Acad. des Sciences de Paris.


Ferry Borges, J. and Castanheta, M. (1968), Structural safety, Laboratorio National de Engenharia Civil, Lisbon, Portugal.
Hanes, B.J. (1956), Limit design and safety factors in soil mechanics, Bulletin No 1, The Danish Geotechnical Institute, Copenhagen, Denmark.
Jonson, A.I. (1953), Strength safety and economic dimensions of structures, Bull. No 12, Royal Institute of Technology, Stockholm, Sweden.
Kanzinczy, G. (1928), Beitrag zum Vortrag Gehlers Internationale Tagung für Brückenbau und Hochbau, Schlussbericht, Wien, Austria.
Maier, M. (1926), Die Sicherheit der Bauwerke und ihre Berechnung nach Grenzkraften anstatt nach zulässigen Spannungen, Springer Verlag, Berlin, Germany.
Matsousek, M. and Schneider, J. (1976), Untersuchungen zur Struktur des Sicherheitsproblems bei Bauwerken, Bericht No. 59, Institut für Baustatik und Konstruktion, ETH, Zurich: Birkhauser, Switzerland.


Müller, G.E. and Schmid, C.J. Factors of safety – historical development, state of the art and future outlook, AGARD Report #661, Air Force Flight Dynamics Laboratory, Ohio, USA.


Revision to BS 449-1937 (1939), British Standards Institution, London, UK.


Whitney, Ch. S. (1929) Bridges; a study in their art, science and evolution, W.E. Rudge, New York, USA.

Essay 2

THE HISTORY OF
THE CONCEPTION OF STRESS
...a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else.

Archimedes

Cauchy's concept has the simplicity of genius. Its deep and thorough originality is fully outlined only against the background of the century of achievement by the brilliant geometers who preceded, treating the special kinds and cases of deformable bodies by complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became and has remained the foundation of the mechanics of gross bodies.

Nothing is harder to surmount than a corpus of true but too special knowledge; to reforge the tradition of his forebears is the greatest originality a man can have..

C. Truesdell
Introduction

After the appearance of the famous book by G. Galilei (Galileo Galilei, *Discorsi e Dimonstrazioni matematiche*, Leiden, 1638) and up to 1820 various partial problems were investigated in the field of mechanics. However, there was one essential circumstance, which had to result in broad generalizations. This circumstance consisted in development of physical theories of the matter structure. In the 18th century Descartes’ idea of all-filling fine matter (plenum) with penetrating “eddies” gave way to Newton’s concept of material bodies composed of the least particles, which interact with the help of central forces. Newton considered his “molecules” as the particles of finite dimensions and certain shape [Newton, 1717], but his followers have gradually reduced them to material points.

The most distinctly expressed theory of this type belongs to Boskovich [Boscovich, 1763], for whom the material points were only permanent centres of forces. The Laplacian capillary theory [Laplace, 1806] and Poisson’s first research [Poisson, 1814] on equilibrium of elastic surface belong to this series of ideas; but probably there were for long no attempts to obtain general equations of equilibrium and motion of an elastic solid.

By the end of 1820 Newton’s conception on the matter structure and Hook’s law had given means for generalizing the principle of possible works in *Mecanique Analytique*, thus opening a broad way for new research both in mechanics and in other fields of mathematical physics. As A. Love [Love, 1906] notes, physical science completed the first period of its development with defined
methods of forming hypotheses and induction, as well as observations and deduction, with a clear aim of studying laws, which interconnect phenomena and with the accumulated stock of analytical methods of research. It was high time for creation of general theories.

One of the possibilities of introducing the idea of stress into the general scheme of abstract notions of theoretical mechanics (Rational Mechanics) consists in its adoption as the basic notion taken from experience. Here the question is in the idea of interaction between two contiguous bodies or two parts of one body separated by an imaginary surface. Physical reality of such action in accordance with this viewpoint is taken as the basis for including this notion into the general scheme. Maybe thus we have to understand the words by Kelvin and Tait [Thomson & Tait, 1867, vol.1, p.220] that force ”is a direct object of sense”. This idea is assumed as a basis of the method used by Euler, when he formulated principles of hydrostatics and hydrodynamics; and it was used by Cauchy [Cauchy, 1827a] in his first works in the theory of elasticity. Following this idea, one should distinguish between two types of forces that is: body forces and surface tractions; the former belong to the number of forces of remote action, while others act under solids collision.

Natural philosophers were not, as a rule, inclined to take both the remote action and that under solids collision as the equal basic notions. They considered that more extensive analysis will allow them to establish identity of the both kinds of action, Sometimes they tried to explain remote action with the help of stresses in the medium. They also tried, on the contrary, to explain stresses, which were considered a result of close-range action, by imagining the central forces acting directly at a distance. Variations in opinions as to these questions were presented in the work by Maxwell [Maxwell, 1890]. Introduction of the system of Maxwell’s stresses equivalent to electrostatic attraction and repulsion [Maxwell,1881] might serve as an example of the first kind aspirations. The method of central forces was used in numerous studies in the theory of elasticity. Cauchy used
this method to determine interrelations between stress and strain components in a crystal body [Cauchy, 1828b] Any such reduction of the short-range action to remote one removes the difference between the surface stress and body forces; they usually tried to maintain this difference by the hypothesis of molecular structure of bodies. In the Cauchy theory the untrue short-range action is reduced to remote action among molecules, in so doing it is taken that this action is not distributed beyond the so-called “region of molecular effect”. Body forces, on the contrary, are considered as those of remote action. Thus the second method of introducing the notion of stress is based on the hypothesis of molecular forces.

The third method is connected with using the notion of energy. It is taken that there exists a potential of elasticity, and the equilibrium equation or oscillation equation of elastic body is derived with the use of variational approach. Let energy of the body part, bounded by any surface $S$, increase under the increase of displacement. This energy augmentation will be expressed by means of the following surface integral:

$$ \iint \left[ \frac{\partial W}{\partial e_{xx}} \cos(x, \nu) + \frac{\partial W}{\partial e_{xy}} \cos(y, \nu) + \frac{\partial W}{\partial e_{xz}} \cos(z, \nu) \right] \delta u + \cdots + \cdots \, dS. $$

Under the energy approach force is determined as the rate of displacement increase in the expression of energy augmentation. The above expression for energy directly points to the existence of forces affecting the surface, bounding any part of the body. In such context the notion of stress becomes a secondary or derivable notion, while energy, difference of its forms and its location in the environment are taken as the basic concepts. This method is limited by the cases of existence of the potential of elasticity.

The first and third approaches are suited more than the second one to such kind theories, which are sometimes called macroscopic like the elasticity theory in its larger part. In the second method they, on the contrary, proceed from molecular, atomistic or subatomic structure of the body. To conform to the purposes of the elasticity...
theory the structural theory should result in the notion of stress and in
the Hook law and existence of the potential of elasticity. Besides, it
should include a possibility that the ratio between the elastic
constants, which we call the Cauchy ratio are not necessarily
preserved. These are four requirements imposed on the theory.

Most structural theories, used in mechanics of rigid body, present
molecules, atoms or elastic elementary particles composing a body as
simple force centers endowed with the mass property. These body
elements affect each other with certain forces, the forces, acting
between two elements $P$ and $P'$, are directed along the line which
connects them and are the reverse of each other. It is usually foreseen
that the forces, acting between structural elements of the body,
disappear when a distance between them exceeds a certain value
called a spherical radius of molecular action. But it is not necessary.
Suffice it to take that these forces decrease so quickly under the
distance increase, that they may be ignored, when distances are small
compared with the least distances, which may be measured using
standard devices.

Navier [Navier, 1827]$^4$ was the first researcher, who was engaged
in constructing equilibrium equations and oscillation equations of the
elastic body. He proceeded from Newton’s conception of the
structure of matter and thought that elastic reactions appear as a
result of those changes of intramolecular forces, which are a result of
changes in the relative position of molecules. He considered
molecules as material points and supposed that the interaction force
of two molecules (the distance between them being something
increased) is proportional to the product of distance increase by a
certain function of initial distance. His method consists in forming
the expressions for projection to arbitrary direction of all forces,
which affect the displaced molecule, and in deriving hence the
equation of molecular motion. The equations obtained in such a way
prove to be expressed in the molecule displacement.

$^4$ The memoir was read in May 1821 that was before its publication.
The material is supposed to be isotropic, and the equations of equilibrium and oscillatory motion include the same constant of the same nature as Young’s modulus. Then Navier forms an expression for the sum of work of all forces affecting a molecule under small displacement; he calls it a sum moment (in a sense of Mecanique Analytique) of all forces applied to a given molecule and caused by all other molecules. Using a variational calculation, he derives not only the already obtained differential equations, but also boundary conditions to be met at the body surface. This issue is very important as the first general research in this problem; however the line of reasoning used was not generally recognized. The objections were advanced against Navier’s expression for the interaction force of two molecules and against his method of simplification of expressions for the forces affecting a separate molecule. These expressions result in the triple summation, which Navier changes by integration; this method validity was called in question.\footnote{Criticism of Navier’s memoir and a statement of discussion that caused it see in [Todhunter & Pearson, 1886, pp. 139, 221, 177]. See also a report [Burkhardt, 1903]. It should be noted that the concept of molecules as material points being at rest in the state of stable equilibrium under forces of mutual attraction and repulsion and somewhat displaced by external forces differs from that conception of molecules, which we know from modern thermodynamics. Molecular theories of Navier, Poisson and Cauchy have little in common with current ideas of molecules.}

In 1821, when Navier had given a report at the Academy, one more field became an unexpected origin of a new powerful impulse for development of the theory of elasticity. Fresnel announced that in his opinion the facts known from observations, which concerned interference of polarized light, may be explained.

\textit{Claude-Louis Marie-Henri Navier}  
\textit{(1785–1836)}

\textit{Augustin-Jean Fresnel}  
\textit{(1788 – 1827)}
only with the help of the hypothesis of transverse oscillations. He showed how such oscillations and propagation of waves of corresponding types occur in the medium consisting of “molecules” connected by the action of central forces. All those examples of transversal waves which were known well before Young and Fresnel, for example, waves on water, transversal oscillations of strings, rods, membranes and plates, presenting the waves propagating within the medium. Maybe both adherents and opponents of the wave theory of light did not even imagine the light waves otherwise than “longitudinal” waves of thickening and rarefaction similar to those which were well known from the theory of sound propagation. Then the theory of elasticity and a problem of wave propagation in the elastic medium attracted attention of two outstanding mathematicians: Cauchy, who was an adherent of ideas of Fresnel, and Poisson – the opponent, who was skeptical of these ideas. Further development of the theory of elasticity was closely related to the problem of light propagation; this development has begun in many respects from the works of these two scientists.

2.1. A. Cauchy generalized stress principle

On September 30, 1822 Augustin Louis Cauchy announced the existence of the stress principle, which became a basis of rational mechanics of continuous media.

6 See [Verdet, 1866] p. LXXXVI, as well as p. 629, and Verdet indicates that Fresnel came to his hypothesis of transversal vibrations in 1816 (citing p. XV, 385, 394), Thomas Young in his paper [Young, 1817] states that light vibrations have only weak transversal components

7 The impetus for Cauchy’s studies in elasticity theory was his participation in the Commission appointed to consider Navier’s memoir on elastic plates, which he presented to the Paris Academy in August 1820. The impetus for Cauchy’s studies in elasticity theory was his participation in the Commission appointed to consider Navier’s memoir on elastic plates, which he presented to the Paris Academy in August 1820

8 Cauchy’s memoir was presented to the Paris Academy in September 1822, but it was not published. Summary was presented in Billet des Sciences de la Societe philomatique, 1823; the memoir content was published in the papers
A. Cauchy set forth this principle in the memoir presented to Paris Academy of Sciences in 1822; its brief content was published in 1823 as a paper [Cauchy, 1823] as well as in a series of next papers. The present definition of the notion of stress was given by A.B. Saint-Venant in 1844 [Saint-Venant, 1844].

The Cauchy principle includes four independent statements:
- physical dimension – “force”/”area”;
- stress is determined at the imaginary boundary, dividing a body in two parts;
- stress is a vector or vector field, equivalent as to the effect of one body part on the other;
- the stress vector direction is not limited.

These four theses appeared and developed independent of each other.

A. Cauchy’s work was prepared by investigations of L. Navier, who had developed molecular theory of the elastic solid body and derived equations of its equilibrium and motion under displacements in the memoir presented to the Paris Academy of Sciences in 1821 and published in 1827 [Navier, 1827] (in a short form in 1823 [Navier, 1823]). Probably, just this work made Cauchy to write the above memoir, since he was appointed a member of the Commission of Navier’s memoir consideration by the Paris Academy of Sciences.

The idea that the elasticity property may be explained by the attraction and repulsion forces, acting between the finest body particles, existed since I. Newton’s time and was a subject of research of R. Boscovich considered in his book [Boscovich, 1763], published in 1763.

[Cauchy, 1827a], [Cauchy, 1827b], [Cauchy, 1828a]. The latter contains rather correctly derived equations.
The principle asserts that at any smooth closed oriented surface \( \partial V \), whether it is imaginary surface within the body or bounding surface of the body itself, there is an integrated field of stress vectors \( \mathbf{t}_{\partial V} \) equipotent to the effect on the part of material, external in respect of \( \partial V \) and contiguous in respect of \( \partial V \) on this inside (Fig. 2.1). Thus the resultant force \( F(V) \) and resultant torsion moment \( L_{x_0}(V) \), that affect the material in the region \( V \), adjacent to \( \partial V \), are determined by equations:

\[
F(V) = \int_{\partial V} \mathbf{t}_{\partial V} \, ds + \int_{V} b \, dm, \\
L_{x_0}(V) = \int_{\partial V} (x - x_0) \wedge \mathbf{t}_{\partial V} \, ds + \int_{V} (x - x_0) \wedge b \, dm, 
\]

where \( ds \) and \( dm \) point to integration in respect of the surface area and mass and where \( b \) is mass force per mass unity. Thus the body is affected by forces and moments of two kinds: those being absolutely continuous mass function, and those being absolutely continuous functions of the boundary surface area. The term body is applied to former, while contact or stress – to the latter. Rational mechanics of materials is the theory of contact forces.

Cauchy admitted that stresses depend on the surface only through a normal:

\[
\mathbf{t}_{\partial V} = \mathbf{t}_n, 
\]

that is why all the bodies, their bounding surfaces having common normal \( n \) in some point, have similar stresses there. Using the principle of motion quantity, he proved then that \( \mathbf{t}_n \) changes a sign, if \( \partial V \) orientation is changed:

\[
\mathbf{t}_n = -\mathbf{t}_n 
\]

or stress, which appears under the inside effect on the outside, has the same or opposite direction as to the stress caused by the effect of the outside on the inside. Using this result, called the Cauchy lemma, and applying again to the principle of motion quantity, Cauchy proves that,
if, by assumption, \( t_n \) is a continuous \( n \) function, then it is a homogeneous linear function \( n \):

\[
    t_n = T_n
\]

(2.5)

That is the linear transformation \( T \) exists in the vector space; it is called a stress tensor, which effect on a single external vector, normal to \( \partial V \) in any point, gives a stress vector in this point. In particular, according to Cauchy himself, the stress on three mutually perpendicular planes in a certain point unambiguously denote stresses on all planes there. The existence of stress tensor is often called the Cauchy basic theorem. Having used it in (2.1), Cauchy proves then that in the point, being inside as to the area, where in a certain time the tensor field \( T \) is continuously differentiated, where \( b \) is continuous and acceleration \( \dot{x} \) exists and it is continuous, the principle of motion quantity in inertial coordinate system is equivalent to partial differential equation.

\[
    \text{div} T + \rho b = \rho \dot{x},
\]

(2.6)

where \( \rho \) – mass density is also, by assumption, continuous. Then he proves that under the same assumption if (2.6) is fulfilled, the principle of angular moment is equivalent

\[
    T = T^T.
\]

(2.7)

Thus the stress tensor is symmetrical. The two equations (2.6) and (2.7) are the first and second Cauchy motion laws. If we confirm validity of the motion quantity and angular moment principles, the Cauchy first law proves that the contact force resultant per volume unit equals \( \text{div} T \), while the second law is a necessary and sufficient condition for the contact torque resultant be a moment of equipotent contact force.

These two laws are insufficiently general for modern mechanics of continuous media. Firstly, the fact that all acting moments are moments of force, as is stated by (2.2), is not always a proper assumption, because there may be contact pairs, and because, in
connection with oriented materials, there may arise a necessity to introduce the higher order stress, motion quantity torque, etc. Secondly, the assumptions of smoothness which are necessary for obtaining the Cauchy laws can prove to be too potent. A long series of studying shock waves and other strong discontinuities, beginning from separate cases considered by Fourier, Poisson, Stokes, Rankine, Hugoniot, Hadamard and Zemplen, has resulted in the Kotschin’s general theorem [Kotschin, 1926], which application to the principle of motion quantity with equipotent force, set by (2.1), gives a condition of jump.

\[
\left[ t \partial_t \nu + \rho U \dot{x} \right] = 0. \tag{2.8}
\]

Square brackets here define a jump over the singular surface in a specific considered place, \( U \) – local velocity of the surface propagation, and \( \dot{x} \) – velocity field. The condition (2.8) is necessary and sufficient for the motion quantity principle with equipotent force set by the Cauchy stress principle be fulfilled in each vicinity of the point, to which it belongs. Thus, this condition does not express a new conception, but it is rather a result of mathematical application of the principle of stresses for the case, when ordinary suppositions about smoothness were too potent. A principle of the moment of motion quantity provided that the equipotent moment set by (2.2) under singular surface is also equivalent to the Cauchy condition.

Thus A. Cauchy has derived three equilibrium equations of elementary tetrahedron, proved the reciprocity law for shearing stresses, introduced a notion of principal axes and principal stresses and derived differential equilibrium equations. He has also introduced a notion of the normal stress surface (Cauchy quadric), where radius vector ends are distributed and their directions coincide with directions of normals for the areas and the value
is inversely proportional to a square root of the absolute value of normal stress in this area, and it is proved that this surface is the second order surface with a centre in the origin of coordinates. A possibility to transform normal stress surfaces to principal axes evidences for the existence of three principal mutually perpendicular areas in each point.

An analogous notion of the surface of tangent stresses was derived by Russian mechanical engineer G.V. Kolosov in 1933 [Kolosov, 1933].

It is of interest that when Cauchy was twelve, great. G.-L. Lagrange noticed his remarkable mathematical abilities, forecasted the boy the great future and gave his parents an advice: “If you would not make haste to give Augustine the well-grounded literary education, he will follow his vocation, become great mathematician but will not be able even to write in his mother tongue. Don’t allow him to touch books in mathematics until he will be nineteen”. The advice was followed. Augustine was educated at school for particularly gifted children, where he studied the humanities, several languages. He was reworded for his verses in French and Latin. Being sixteen he entered Ecole Polytechnique and then Ecole des Ponts et Chaussees. After graduation from it a young engineer worked in Cherbourg being engaged in construction of port and defense structures. Simultaneously he started scientific activity connected with design of stone bridges, arches and other structural forms. But a young researcher was attracted by scientific problems with the use of mathematics. Cauchy has written more than 700 mathematical works, where he laid foundations of current mathematics.

Geometrical interpretation of a stresses state in space in the form of stress ellipsoid was proposed by G. Lamé and B. Clapeyron in their memoirs presented to the Paris Academy in 1828 and published in 1833 [Lamé & Clapeyron, 1833].

Geometrical presentation of the stressed state on a plane for one series of areas, which pass through the principal axis in a form of stress circle was proposed by Culmann in his book in 1866 [Culmann, 1866].
Geometrical interpretation on the plane given by O. Mohr (so-called Mohr’s circle diagram) [Mohr, 1822] in 1882 for a general case of stress state is very obvious. Proceeding from this interpretation one can make a number of important conclusions on extremality of principal stresses, position of sites on which tangent stresses are maximum, as well as on the values of these maximum tangent stresses.

A. Cauchy has defined deformations, derived their dependence on displacements in a separate case of small deformations, determined a notion of principal stresses and principal deformations and obtained dependences of stress components and deformation components both for isotropic and for anisotropic elastic body. They are called a generalized Hook law, though this name is conventional because Hook was not acquainted with the notion of stress.

In the above dependences Cauchy first introduced two constants and recorded stress dependences in the form

\[ \sigma_x = k\varepsilon_x + K\Theta, \quad \sigma_y = k\varepsilon_y + K\Theta, \quad \sigma_z = k\varepsilon_z + K\Theta, \]

\[ \tau_{xy} = k \frac{\gamma_{xy}}{2}, \quad \tau_{yz} = k \frac{\gamma_{yz}}{2}, \quad \tau_{zx} = k \frac{\gamma_{zx}}{2}. \]

There \( \Theta = \varepsilon_x + \varepsilon_y + \varepsilon_z \), that is, as in the theory of elasticity (with other marks). But then A. Cauchy accepted L. Navier’s conception. Following the latter elastic bodies consist of molecules; forces which appear among them under deformation act in the directions of straight lines, connecting molecules, and these forces are proportional to the change of distances between molecules. Then the number of elastic constants for a general case of anisotropic body is equal to 15, and for the isotropic body we obtain one elastic constant. G. Lamé and B. Clapeyron formerly kept to this hypothesis and then S.-D. Poisson joined them.

In 1839 G. Green derived a dependence between deformations and stresses without using the hypothesis of molecular structure of elastic bodies [Green, 1839]. He obtained them on the basis of the energy conservation principle, introducing a conception of elastic potential, and demonstrated that, when using linear dependences of
six deformation components on six stress components, 21 coefficients of 36 are independent, i.e. in the general case of anisotropic body the number of constants equals 21. The number of elastic components for the anisotropic body decreases to two.

As A.E.H. Love notes [Love, 1935], the works by Green may be compared only with Navier’s basic equations as to their significance for the foundations of the elasticity theory. Proceeding from the law known now as the “principle of energy conservation”, he proposed a new method for obtaining these equations. He himself stated his principle and method as follows:

“In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. But this function being known, we can immediately apply the general method given in Mécanique Analytique, and which appears to be more especially applicable to problems that relate to the motions of systems composed of an immense number of particles mutually acting upon each other. One of the advantages of this method, of great importance, is that we are necessarily led by the mere process of the calculation, and with little care on our part, to all the equations and conditions which are requisite and sufficient for the complete solution of any problem to which it may be applied.” [Love, 1906].

The above function is potential energy of a deformed elastic body taken with opposite sign, referred to the volume unit and expressed in deformation components; as to deformation components, this function partial derivatives are equal to stress components. Green supposed that this function could be expanded in terms of powers and products of deformation components; thus he presented it as a sum of homogeneous functions of these values of the first, second, third and higher orders. The first of these terms was to be equal to zero, since potential energy before deformation would have the least value; and since all the deformations are small, only second term is significant.
Green has derived from this principle his elasticity theory equations, which contain 21 constants in the general case. Only two constants remain in the case of isotropy, and equations coincide with those presented in the first Cauchy’s memoir.

It is of interest that George Green (1793-1841) had not been educated in his youth, studied mathematics himself and only in 1833 entered Cambridge University, which he graduated from in 1837, i.e. at the age of 44. He developed the theory of electricity and magnetism in his essay *Application of Mathematical Analysis to the Theory of Electricity and Magnetism* based on the concept of potential and the ratio between the volume integral and surface integral limiting this volume. M.V. Ostrogradsky obtained the above ratio in the same year independent of Green. In 1839 G. Green, based of the principle of energy conservation, derived dependences between deformations and stresses for the elastic anisotropic body.

Lord Kelvin presented a proof of existence of Green’s elastic potential based on the first and second laws of thermodynamics. Using these laws, he makes a conclusion that, when the rigid body deformation is not accompanied by the temperature change, the stress components are partial derivatives of some functions of deformation by above components of deformation. It may be proved that it is also true in the case, when deformation occurs so quickly, that neither heat absorption nor transfer take place in any part of the body.

*Siméon-Denis Poisson (1781–1840)*

*George Green (1793–1841)*

*William Thomson, 1st Baron Kelvin (1824–1907)*
The methods of Navier, Poisson and Later Cauchy’s memoirs result in the motion equations, which contain a less number of elastic constants than the equations obtained by the methods of Green, Stokes and first Cauchy memoir. The significance of this diversity was first emphasized by Stokes. It was debated whether elastic properties of the isotropic body were defined by two or one constant. Pearson called these two theories “multiconstant” and “rariconstant” ones [Todhunter & Pearson, 1886, p. 496]; polemics as to these theories was too long. Rariconstant equations may be obtained from multiconstant ones, if some pairs of coefficients be equated in the latter, but rariconstant equations are based on a certain special hypothesis of the matter structure, while the acceptance of multiconstant theory is connected, as it was thought, with this hypothesis negation. The discrepancy between results of the both theories could be removed by means of experimental research, and the question would be definitely settled; but numerous experimental studies are depreciated by the fact that it is too difficult to state with certainty the material isotropy, and a tendency of many adherents of multiconstant theory to be based on testing such materials as cork, gelatine, rubber only made their argumentation weaker. Most discussions concerned the value of the ratio of lateral rod shortening to its longitudinal elongation under the effect of the extension load applied to its ends. This ratio is called Poisson’s ratio. Based on his theory Poisson came to a conclusion that this ratio was 1:4. The experiments of Wertheim [Wertheim, 1848] with glass and brass had not confirmed these results, and Wertheim offered to consider it to be equal to 1:3 – the value which had no theoretical substantiation. Based on experimental material Lamé [Lamé, 1852] arrived in his treatise to multiconstant equations, which became conventional after the book had been published. Saint-Venant, who was a convinced adherent of the rariconstant theory, expounded results of his studies of torsion [Saint-Venant, 1855], bending [Saint-Venant, 1856] and distribution of elastic properties in a certain given point [Saint-Venant, 1863], using the multiconstant theory. Kirchhoff [Kirchhoff,
1850, 1859a] used the same theory in his research of thin bars and plates and confirmed it by experiments on torsion and bending of steel bars [Kirchhoff, 1859a]. Clebsch in his treatise [Clebsch, 1862] also used terminology of contactless isotropy. Kelvin and Tait [Thomson & Tait] did not almost mention this discussion and joined Stokes’ views. Experiments confirm the conclusion that the Poisson ratio can essentially differ from 1:4 for materials, which may be considered as isotropic and homogeneous with no stretch.

However the most astonishing data were obtained by Voigt [Voigt, 1887, 1910], when he studied elastic properties of crystals. The absence of confidence in isotropy of materials subject to tests ceased to be an obstacle after he had risked to make experiments with wittingly anisotropic material⁹. At the same time the difficulties to be settled were more extensive. According to Green the material with the most general anisotropy is characterized by 21 independent constants. Molecular hypothesis developed by Cauchy and maintained by Saint-Venant results in 15 elastic constants, i.e., if the rariconstant theory is correct, 21 Green’s constants are to be connected by 6 independent ratios¹⁰. Voigt’s experiments consisted in torsion and bending tests of prisms of various crystalline materials; Saint-Venant’s formulas for anisotropic rods are valid for most of them, Voigt found necessary formulas for others; Cauchy’s ratios were confirmed with a certain extent of accuracy only for beryl and rock salt; it was found out for 7 other tested

---

⁹ The known proposition first made by F. Neumann consists in the statement that crystal anisotropy as to elasticity may be studied with the help of investigation of crystallographic forms.

¹⁰ They were probably first obtained by Saint-Venant in his work on torsion [Saint-Venant, 1855], though A. Lyav [Lyav, 1935] calls them Cauchy’s ratios.
crystalline materials that the coefficients, which were to be equal, according to these ratios, essentially differed from one another.

Independent of experimental data the rariconstant theory lost its significance because of the extension of knowledge in matter structure. This evolution of knowledge in physics is determined by numerous causes, the divergence of rariconstant theory plays a comparatively subjected part among them. Development of atomistic theory in chemistry, statistical molecular theory in physics, propagation of energy principles and discovery of electromagnetic radiation were of much higher importance.

2.2. Main stages of the history of A. Cauchy’s generalized principle

The following main stages of the history of Cauchy’s generalized principle may be distinguished:

- antiquity – Democritus, Archytas, Aristotle, Archimedes;
- renaissance – Leonardo da Vinci;
- works by Galilei in tensile rupture of bodies;
- works by Hooke, Mariotte, Young;
- concept of tension of flexible filaments – G. Pardis (1673), J. Hermann (1716), Jacob Bernoulli (1744). Studies in elasticity by L. Euler (1727);
- principle of solidification by S. Steven (1586). Works in hydrostatics, hydraulics, hydrodynamics – I. Newton (1687), B. Taylor (1715), L. Euler (1740), Daniel Bernoulli (1730), Johann Bernoulli (1740), L. Euler (1749-1752), G. D’alembert (1749), L. Euler (1750-1766);
- beam shearing stress – G. Galilei, E. Mariotte, H. Leibnitz (1684), A. Paran (1713), Varignon, Coulomb, Jacob Bernoulli, Euler, Dulot, Tredgold;
- determination of shearing stresses – Coulomb (1773);
- allowance for shearing under beam bending. D.I. Zhuravsky;
- generalized stress theory – Descartes, Newton, Boscovic, Navier, Cauchy, Fresnel, Poisson, G. Green, Kelvin, Stokes, B. Saint-Venant, Voigt.

2.2.1. Antiquity

First knowledge in the field of mechanics before the 6th century B.C. were obtained (using current terminology) in hydraulics, structural mechanics, static, dynamics and celestial mechanics.

Democritus (about 460-370 B.C.), a well-known Greek materialist, had developed the study of motion. He taught that the matter consists of atoms, fine particles which do not split, are of different size and shape. Atoms move in space in various directions and with various velocities without acceleration and deceleration, and thus uninterruptedly. This motion is eternal, it has neither the beginning nor the end. So, Democritus had anticipated the law of inertia; the difference was that he admitted not only linear, but also circular motion of atoms.

It is considered that Archytas of Tarentum (about 428-347 B.C.), a representative of the school of Pythagoras, was the first, who had applied mathematical methods to systematized studying of mechanics.

Mathematics of the period of Hellenism mainly developed in applied trend: construction, creation of new machines, high consideration of the strength of structures, etc.

The principle of virtual velocities (displacements) for a lever was formulated even in Physics by Aristotle (384-322 B.C.), an outstanding Old-Greece philosopher. Therewith, Aristotle, as L. Euler [Euler, 1959] noted, had borrowed this thought from his predecessors.

A developed conception of force or stress level (by the way, it was completely formulated 2000 years later) is presented in Aristotle’s works Physics and Treatise on the Heaven. This force appeared in Aristotle as the prime cause of motion. He knew of the composition of motions by parallelogram rule, as well as the notions of velocity and resistance of medium.
In the work *Mechanical Problems*, which belonged to one of Aristotle’s pupils (this work was attributed to Aristotle, thus its author is called now *Pseudo-Aristotle*) and was written in the 3rd century B.C. we even find the definition of mechanics (Greek word *mechane* means artfulness). This work included first explanation of equilibrium of bodies, and thus, it became a fundamental one for creation of static. It showed that human artfulness can control nature, fight “against nature”; Pseudo-Aristotle writes: “Nature, so far as our benefit is concerned, often works just the opposite to it. For nature always has the same bent, simple, while use gets complex. So whenever it is necessary to do something counter to nature, it presents perplexity on account of the difficulty, and art [techne] is required. We call that part of art solving such perplexity a mechane.”

The problem which caused rather great disputes in the 17 and 18th centuries – the problem of difference of the pressure and impact action is also considered among others in *Mechanical Problems*. “Why, if you put a large ax on wood, and a large burden on that, doesn’t it pull apart the wood, no matter how considerable the burden is? But if you raise the ax and hit with it, you split the wood even if you have less weight than you put on the ax in the first place?” However, Pseudo-Aristotle could not give a satisfactory answer to the question – this question required the extensive study of motion transformation. Note, that already in the work *On the Heavens* Aristotle presented the value $mv$ as the force dimension.

Archimedes (287-212 B.C.) was a famous mathematician and mechanician of antiquity. He has made great contribution to mathematics, mechanics, physics and astronomy, approached creation of the integral calculus. He has also found out the principle of centre of gravity, created a strict system of static, laid grounds of hydrostatics. He has made a lot of inventions in the field of applied mechanics and created a lot of new mechanisms.

Mathematical genius of Archimedes had been exhibited by the fact that he began solving the most difficult problems of that time:
calculation of the areas of curvilinear figures, calculation of surfaces and cylinder and sphere volumes. These problems brought him (in the work *Ephodikon*, discovered in 1906) to establishing the basic notions of integration. It had taken about 1700 years for I. Newton and G. Leibnitz actually founded differential and integral calculus. Archimedes also displayed his mathematical genius in solving mechanical problems. His main achievements: the law of lever and Archimedes’ law were obtained by geometrical method. We can rightfully call Archimedes a pioneer of mathematical physics. Archimedes’ static is expounded in treatises *On Equilibrium of Planes* and *On Floating Bodies*. The law of lever is described in the first treatise. The treatise central idea was the notion of the centre of gravity. Empirical data on equilibrium of a heavy body had long been known. Even Egyptians used the plumb-line. But only Archimedes had a distinct idea of such a point inside a body, the weights of all its other points being balanced in respect of this point in such a way that the body leaned on this point be balanced.

![Democritus](image1)
Democritus
(aapr. 460–370 B.C.)
Old Greek
Δημόκριτος

![Archytas of Tarentum](image2)
Archytas of Tarentum
(428–365 B.C.)
Old Greek
Αρχίτας

![Aristotle](image3)
Aristotle
(384-322 B.C.)
Old-Greek
Ἀριστοτέλης

![Archimedes](image4)
Archimedes
(appr. 287-212 B.C.)
Old-Greek
Ἀρχιμήδης

Now address to the other Archimedes’ result, his illustrious law [Kudryavtsev, 1948]. There is a well-known Vitruvius’ story how Archimedes tested his discovery. It is out of doubt that the experiment suggested Archimedes the idea and gave him an
opportunity to test it. Moreover, Archimedes, undoubtedly, could determine specific weight by experiment; some people even mention a float which helps to compare specific weights of fluid (aerometer). But being true to his method, Archimedes wants to prove law with the help of mathematics, proceeding from some postulates. Archimedes assumes the hypothesis of liquid nature as a basis: “Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else”\(^\text{11}\).

Proceeding from this hypothesis Archimedes shows that the liquid surface at rest should be a sphere, which centre coincides with the Earth centre. Really, if it was otherwise, there were no equilibrium: some parts of liquid would be compressed more than others, and this, according to postulate, would result in the displacement of less compressed particles.

This Archimedes’ theorem plays the leading part. Hence, he, first of all, proves that the bodies with the same specific weight as liquid (“solids, those which, size for size, are of equal weight with a fluid”) sink to such extent that do not absolutely emerge to its surface, but also do not sink something deeper.

Then Archimedes formulates his law in the following two statements:

“Proposition VI. If a solid lighter than a fluid is forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.

Proposition VII. A solid heavier than a fluid will, when placed in it, descend to the bottom of the fluid, and the solid will, when

\(^{11}\) Citation was taken from the book: Archimedes, Stevin Simon, Galilei Gallileo, Pascal Blaise. Origins of Hydrostatics. – Moscow-Leningrad, GTTI, 1933, 404 p.
weighted in the fluid, be lighter than its true weight by the weight of the fluid displaced.”

Under Archimedes mechanics of antiquity reached its culmination point. The followers had not add anything (except for Pappus’ researches on gravity centre); they were lost in the Middle Ages, and Archimedes’ study on bodies buoyancy was substituted by the scholastic study that the floating of bodies is determined by their shape.

2.2.2. Renaissance. Leonardo da Vinci

Renaissance had inspired interest to mechanics as science. Leonardo da Vinci was the most significant representative of that epoch. As Truesdell [Truesdell, 1968] notes, only one of Leonardo’s rules is valid nowadays, and it is too evident: the rod strength is proportional to its cross-section area.

Leonardo studied experimentally the strength of building materials, studied bending of beams, in particular, the effect of their sizes on strength, support of columns; he established that their carrying capacity is in inverse proportion to length and in direct proportion to their cross-section area.

Kurrer [Kurrer, 2008] notes that one can find in Leonardo’s drawings numerous examples of principles of the theory of structures and material resistance that exceed, as to their importance, great contributions of antiquity scientists of Hellenism period, such as Archimedes, Heron, Ktesibios, and are closely connected with technical meditations of Leonardo. Being more known as a painter, author of Mona Lisa and The Last Supper, Leonardo-engineer presented first written evidences of experiments in the field of material resistance in his notes dated from 1500, known to historians in culture, science and engineering under the title of Codex Atlanticus.
2.2.3. Galilei

Galileo Galilei was the first in the history of mankind who had raised a question of the strength of bodies and the first who tried to answer it. Only Leonardo da Vinci was engaged in the problem of strength and stability before Galilei. But his works had not been published and thus had no influence on development of the science of strength.

Galileo di Vincenzo Bonaiuti de 'Galilei was Italian physicist, mechanician, mathematician, astronomer, one of the founders of exact natural history. Galilei was born in the town of Pisa [Fahie, 1903] (Italy) and descended from the noble Florentine family. He was primarily educated at Valdambroso monastery near Florence, where he studied Latin and Greek, as well as logic. In 1581 he was admitted to Pisa University, where he had to study medicine. But he soon took a great interest in lectures in mathematics and was deep absorbed in studying works by Euclid and Archimedes. Maybe he got acquainted with discoveries of Leonardo da Vinci in the field of mechanics, when read works by Cardano (see [Duhem, 1905, p.39])\(^\text{12}\). In 1581 he made a hydrostatic balance to measure density of various substances and made experiments on finding gravity centres of rigid bodies. This work\(^\text{13}\) made him popular and in mid-1589 he began reading mathematics at the Pisa University as professor. He was not even 26.

During his stay in Pisa (1589-1692) Galilei continued researches in the field of mathematics and mechanics, in particular, performed his experiments with falling down bodies. Based on these

\(^{12}\) G. Cardano discusses questions of mechanics in one of his mathematical works. His notions as to this science are similar to those of Leonardo da Vinci, but it is not usually allowed for that Cardano was acquainted with manuscripts and notebooks of the latter

\(^{13}\) Published in 1586 under the title *La Bilancetta* (Small Balance).
experiments he had written a treatise *De motu gravium* (1590) that was the beginning of dynamics which we know. This work conclusions were as follows: 1) all bodies fall from the same altitude during equal time intervals; 2) velocities of bodies at the end of their fall are proportional to falling duration; 3) the body fall distance is proportional to the square of fall time. These conclusions differed from the principles of Aristotle’s mechanics, but Galilei doubtlessly leaned upon them in his disputes with representatives of Aristotle’s school. That caused animosity against young Gallilei, that made him eventually to leave Pisa and come back to Florence. In these hard times Gallilei’s friends helped him obtain a professor’s position at the University of Padua.

On December 7, 1592, Galilei entered upon his duties with a speech, which compelled unbelievable admiration not only because of its deep erudition, but also by its eloquence and exquisite formulations. Galilei was exceptionally active during his first years in Padua. His lectures became so popular, that students from other European countries came to attend them. Sometimes these lectures were held at a lecture-hall which contained 2000 students. In 1594 he wrote his illustrious treatise in mechanics “Delia scienza meccanica” (*On Science Mechanics*). Various problems of statics were solved in the treatise with the help of the principle of virtual displacements. Therewith, in connection with some shipbuilding problems Galilei had taken an interest in material resistance [Timoshenko, 1957]. But soon he turned his attention to astronomy. It is known that in the first years of his pedagogical activity in Padua Galilei kept to Ptolemaic system, as it was usual then. But already in 1597 he writes in his letter to Keppler: “Many years ago I began to incline to Copernicus ideas, and in the light of his theory I could explain a lot of such
phenomena, which looked absolutely unexplainable on the basis of old hypothesis”.

Galilei was a follower of M. Copernicus (1473-1543) and searched for corroboration of correctness of his teaching on heliocentric system of the world. But in 1616 Copernicus teaching was declared to be heretical. After peripetias with inquisition and “voluntary” renunciation (in 1633, after interrogations, under the threat of tortures) of the cosmogonic theory, Galilei settled in Arcetri near Florence. There “the grand heretic”, who was already 69, devoted his active spirit to the long-planned (though far from being safe in that time of trouble) work in physics, mathematics and mechanics. There he had created his great work Dialogues and Mathematical Proofs Concerning Two New Sciences Pertaining Mechanics and Local Motion. Galilei’s interest in the problems of strength was natural. The great astronomer was no less gifted engineer. His desert was that he had initiated the development of two new sections of mechanics – dynamics and material resistance as independent sciences. Lagrange stated that Galilei had laid first foundations of dynamics. Galilei himself has reduced a great range of questions, connected with strength and break of materials, to one sphere of knowledge (now it is called material resistance).

In 1638 Gallilei, who spent the remainder of his life in the villa Arcetri near Florence, received copies of the book published in Italian «Discorsi e dimostrazioni matematiche intorno a due nuove scienze...» (Dialogues and Mathematical Proofs Pertaining Two New Sciences... ) just printed at Elsevier’s printing house in distant Dutch town of Leiden. It was difficult to find a publisher in Europe, who consented to print a new work by a scientist convicted by inquisition. The 74-year old Galilei took a book, touched it and put aside. He could not see his last book – he was blind.
In this book, written in the form of dialogues, Galilei expounded fundamental studies in material mechanics and dynamics. The first two dialogues are dedicated to the principles of material resistance and structural mechanics.

As S.P. Timoshenko [Timoshenko, 1957] writes, a part of the book dedicated to mechanical properties of building materials and investigation of strength of the beams is the first published work in the field of material resistance; the history of mechanics of elastic bodies starts since its publication.

All Galilei’s works in mechanics of materials were included in the first two dialogues of his book on two sciences. He begins his presentation with reference to some observations made during his visit to Venetian ordnance depot and consideration of properties of geometrically similar buildings. He asserts that geometrically constructed buildings become more and more unstable as their absolute dimensions increase. To explain his thought he indicates: “A small obelisk or column or other solid figure can certainly be laid down or set up without danger of breaking, while very large ones will go to pieces under the slightest provocation, and that purely on account of their own weight.” To confirm the foregoing he begins studying strength of materials in simple tension (Fig. 2.2) and determines that the beam strength is proportional to its cross-section area and independent of its length. Galilei calls such strength of the beam “absolute resistance to fracture” and presents several numerical values that characterize copper strength.

Salviati (in Galilei’s book) is going to refute a common error that a long rope cannot endure the same big weight as a short one (Fig. 2.3). There is a particular rope; he thinks that a certain weight fixed at point C is sufficient to tear it up, and he asks Simplicio where will it be torn apart. Simplicio chooses point D, “since a rope is insufficiently strong at this point to endure, e.g. weight of 100 pounds”. Then Salviati offers to imagine that the rope is fixed at point F, somewhat above D, and the load is fixed at point E located somewhat below D. He says that the rope is still subject to the same
tension at $D$, thus, a short fragment $FE$ will tear up again at point $D$ according to Simplicio’s assumption. This *reductio ad absurdum* (carrying to absurdity) is a typical example of Galileis method of expounding his assumptions in the form of rhetoric [Truesdell, 1968]. In reality he admitted that the effect of the rope and load $DC$ on the rope part above point $D$ is reduced to a certain force acting in the direction of $DE$. In other words, the effect of the system distributed below $D$ on the system distributed above $D$ is equivalent to a certain force applied at the point $D$.

![Fig. 2.3. Proof of Galilei that a long rope is not weaker than a short one](image1)

![Fig. 2.4. Some Galilei’s thoughts on the breaking strength of a cantilever beam](image2)

The proof itself looks rather naïve, since it proves that the rope will either never tear or tear at once and everywhere. The conception of critical loading does not explain a tear of one rope, and, in accordance with everyone’s experience, a long length of cable wound on a bobbin usually, though not always, tears up under a less load than a short one taken arbitrarily from the same bobbin.

When Galilei considers the tensile break of cylindrical beams, he proves that force required for this break should be the same as the basement area, since quantity of fibers, threads or other components which retain the rigid body parts increases with the area. That is he
admits that the tensile break occurs when the force per area unit reaches a certain value.

2.2.4. Hooke, Mariotte, Young

The well-known scientist-encyclopedist Robert Hooke (1635-1703), whose scientific creative work embraces numerous sections of natural science, formulated in 1660 and recorded in his cryptogram in 1676 the following statement “The extension is such as the load” (and not vice versa). It has gone down in history as Hooke’s law and has become a basis of the following development of mechanics of elastic bodies. When creating a clock exact rate controller, Hooke tested flat spiral springs and established that the spring twist angle is proportional to the applied moment. Then he repeated experiments with extended twisted spring, extended steel wire, cantilever woody beam, bent with the force applied to free end. In the course of these studies he has established that displacements are in all cases proportional to applied forces [Hooke, 1931]. Thus Hooke’s law was obtained experimentally for the following types of loading: tension (steel wire), torsion (twisted spring), bending (spiral spring and woody beam).

It is of interest that both Hooke’s life was full of secrets and the appearance of his outstanding law was mysterious. The law was published in Hooke’s book in the form of cryptogram “ceiinosssttvu”

Fourteen Latin letters are arranged in alphabetical order. Being ordered in the way known only to the cryptogram author, they serve to start the words of the phrase revealing the law essence: «Ut tensio, sic vis» that is, the extension is such as the force.

The Hooke’s law is so important because it is general. A. Poincare, the well-known French mathematician, physicist and philosopher, wrote in his book Science and Method : “… each law is the more valuable, the more general it is”. It should be added that the more interesting and general is the scientific law, the more
laconic is the form of its recording, and the simpler it is itself. Hooke’s law corresponds completely to this view.

Only a shortened list of Hooke’s discoveries and inventions in different fields of engineering takes a lot of place.

In 1665 Hooke first described the structure of some plant tissues, in particular cork, consisting of small chambers bound by septa. Thus he has discovered a cell. The science dealing with cells, called cytology, had been formed by the efforts of numerous scientists mainly in the 19th and first half of the 20th century.

In the same year R. Hooke issued his classical work Micrography, dedicated to optics and microscopy (by the way, he had perfected a microscope). Here he gave, in particular, the results of studying the plant structure and introduced such a term as cell.

Hooke was recognized as a good architect. After the great fire of London in 1666 he was a chief assistant of Christopher Wren in reconstruction of the city. He had built as the architect, jointly with Wren and himself, a lot of buildings (e.g. Greenwich Observatory, Willen Church in Milton Keynes), collaborated with Wren in constructing the St. Paul Cathedral of London; the cathedral dome was built by Hooke’s method. He also offered a new planning scheme of London reconstruction.

R. Hooke was also concerned with the wave theory of light, described the diffraction of light and a number of other light phenomena. He performed experiments with woody cantilever beams, measuring their bends, and came to a conclusion that fibers on the convex side are extended, on the concave – compressed. In 1666 Hooke made a scientific report at the Royal Society, he said: “I will explain a system of the world very different from any yet conceived…”

In essence, he has substantiated the law of universal gravity. In 1668 he wrote in his letter to Newton that “force, which controls planetary motion, changes depending, to some extent, on the distance”. But Hooke’s sagacious ideas remained unrealized. The priority of discovery of the law of gravity belongs to I. Newton (1683). There was a notorious action of R. Hook against I. Newton
for the law authorship. At last I. Newton made reference to R. Hooke, and the question was settled.

V. I. Arnold in his book *Huygens and Barrow: Newton and Hooke* argues (also with documents) the statement that it was Hooke, who had deduced the universe gravity law (the method of least squares for the central gravity force) and even substantiated it rather correctly for the case of circular orbits. I. Newton developed this substantiation for the case of elliptical orbits (Hook, on his own initiative, informed Newton about his results and asked him to study this problem). Newton’s quotations cited in the book evidence that Newton considered his part of the deduction as much more important one (because of its complexity, etc.), but he did not object that the law had been formulated by R. Hooke. Thus the priority of formulation and permanent substantiation should be given R. Hooke, and he, probably had formulated the task to complete the substantiation for I. Newton. The latter stated that he had already made this discovery himself but had not left any documentary evidence and informed nobody about the discovery. Besides, I. Newton stopped the works over this problem, and renewed them under the influence of R. Hook.

The presented fact was not the only R. Hooke’s conflict apropos his priority. It is known that he contested his rights with K. Huygens and other scientists that is explained by versatility of Hooke’s interests and passions. As a result he intruded into the sphere of scientific studies of other researchers. He achieved a success in a short term, then his interest cooled down, and he often had no time and patience to complete his work.

And the last interesting feature of his character: he was a very sociable person and met different people. One could often see him in the port. As a result of his conversations with sailors he made a report in 1696 at the Royal Society about a new map of “Tartaria” – the vast territory including the Urals and western part of Siberia.

E. Mariotte, a French physicist, has formulated the law called by Hookes name almost simultaneously with R. Hooke (1680) and
independent of him. The law is as follows: “Even the most rigid bodies – glass and iron – are deformed in proportion to load”. That is, \( f = kP \), where \( P \) – load, \( f \) – rod deformation, \( k \) – proportionality factor.

Edme Mariotte (1620-1684) was a French physicist, one of founders (1666) and first members of the Academy of Sciences founded in Paris. He was born in Dijon, served as a prior of St. Martin monastery in St. Martin sous Beaune in the vicinity of Dijon [Khramov, 1983, p. 179].

His scientific works belong to mechanics, thermodynamics and optics. In 1676 he discovered the law of dependence of the volume of ideal gas on pressure at a constant temperature (it is often referred to as the Boyle-Mariotte law, since this law was found and published by R. Boyle in 1662). Mariotte provided for different applications of this law, in particular, calculation of locality height by barometer indications. He described numerous experiments in the study of fluid flow in pipes, confirmed experimentally conclusions made by G. Gallilei and E. Torricelli concerning fluid leakage rate. Mariotte studied water elevation height in fountains and composed a table of water elevation height dependence on outlet diameter. The scientist studied deformations of elastic bodies, pendulum fluctuations. Mariotte generalized studies in this sphere in his *Treatise on Impact and Collision of Bodies* (1678), generalized research in this field. In 1666 he found a blind spot in a human eye. He studied colors, in particular, he was the first to describe colored halos around the Sun and the Moon, studied a rainbow, ray heat, showed difference between light and heat rays. Mariotte died in 1684 in Paris.

Proportionality factor \( k \) was deciphered 130 years later, when English physicist Thomas Young (1773-1829) first introduced a notion of elasticity modulus \( E \) (referred to as the Young modulus), when investigating tension-compression. Then it could be recorded

\[
 f = \Delta l = \frac{Pl}{EF}, \quad \text{where} \quad L \quad \text{the rod length,} \quad F \quad \text{cross-section area,} \quad E \quad \text{Young modulus,} \quad P \quad \text{load (force).}
\]
Interestingly, that T. Young was a physician, researcher in the field of medicine and physiology. His works also concerned optics, acoustics, heat, mechanics, astronomy, geodesy, philology. He was one of the founders of the light wave theory, explained light interference, etc.

People’s paths to science are different. T. Young belonged to researchers by nature. He read at two, began his self-education at six, and beginning from 9 studied mathematics and languages. He knew French, Italian, Jewish, Farsi and Arabic at fourteen, and started to give lessons. At fifteen he began working over *Philosophical Treatise*. He studied medicine from 19 at three Universities simultaneously. Medicine became his life-work. He presented to Royal Society his work *Theory of Vision* and soon *Principles and Experiments on Sound and Light*. A doctor’s degree in medicine was conferred on Young, when he was 23. In 1807 he published two volumes in natural philosophy, 900 pages each. He had profoundly studied Egyptian hieroglyphs and the theory of music. Young played all the known musical instruments, even Scottish bagpipes. He understood painting. In so doing he always worked as a medical practician.

Many wonderful things have been written about Young, but we should add something to complete his unique image. In his childhood, being absorbed in botany, Young made a microscope for his experiments. As far as he was too judicious, he had studied differential calculus for calculations and lathe work for operations.
As he studied at Universities, he appeared in the circus as a equestrian, trick-rider on two horses and rope-dancer no worse than a professional circus actor.

A versatility of Young’s gifts allows many researchers to compare him with Leonardo da Vinci. T. Young had developed a personal rule: “Each person is able to do all that other people do”, and he followed it all his life. He was simultaneously engaged in different affairs and surely achieved success.

T. Young developed the theory of beam bending, theory of impact fracture of rigid bodies. He was the first to introduce the term energy. Note, that the term work was introduced by G.-G. de Coriolis.

T. Young established the limited character of Hooke’s law, i.e. its validity only at the initial stage of loading, as well as determined the idea of elasticity modulus, though in the form which differed from the then form. T. Young has introduced two values: the modulus weight $EA$, where $A$ – the bar cross-section area, and the modulus height $\frac{E}{g \cdot \rho}$, where $\rho$ – body material density. The first value is not the material constant. That is tensile rigidity of the bar. The other – the material constant with length dimensionality. Young defined the elasticity modulus in the following rather vague form: “The modulus of the elasticity of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight causing a certain degree of compression as the length of the substance is to the diminution of its length.”

T. Young noticed that dimensions of the transversal bar change under tension-compression. These propositions were formulated by him in a 2 volume course of lectures published in 1807 [Young, 1807], which Young read at the Royal Institute.

T. Young also uses such expressions as modulus weight and modulus height, noting that the modulus height for a given material does not depend on the cross-section area. The modulus weight equals the product of the value, called now the Young modulus, by
the bar cross-section area. T. Young defined the steel elasticity modulus value, when observed the camertone vibration frequency, and found that it was equal to $29 \cdot 10^6 \text{ psi} = 2 \cdot 10^5 \text{ MPa}$.

K. Truesdell [Truesdell, 1968], a specialist and historian in mechanics, marks that the concept of elasticity modulus $E$ is present in the manuscript by L. Euler written in 1727 (80 years before publication of T. Young’s book) but published only in 1862, though for its farther use he took the value $\frac{E}{g \cdot \rho}$, i.e. the modulus height according to Young [Bell, 1984].

Giordano Riccati (1709-1790), mathematician, mechanicist and architect, measured in experiment the frequency of bending vibrations of steel and brass cylinders and determined in 1767 the ratio of their elasticity moduli [Riccati, 1782], i.e. he had made the first experimental investigation of elasticity moduli.

The absolute value of the ratio of transversal and longitudinal deformation, a constant within the limits of Hooke’s law validity, is connected with the name of S.-D. Poisson, who introduced it in his memoir presented to the Paris Academy of Sciences in 1829 [Poisson, 1829] and established on the basis of the rariconstant molecular theory that it equals $1/4$.

It should be noted that B. Saint-Venant principle is usually formulated, when posing the problem on bar extension. After this principle relating to bars the peculiarities of application of external forces to the extended or compressed bar display, as a rule, at a distance which do not exceed typical bar cross-section dimensions. This principle was expressed by B. Saint-Venant in his memoirs [Saint-Venant, 1855, 1856] and substantiated by means of experiments. This principle was extended for a general case of deformed body by Boussinesq (1842-1929), a follower of Saint-Venant in 1885 [Boussinesq, 1885].

---

14 Translation of these Memoirs into Russian: [Saint-Venant, 1961].
The works with a proof of the principle are considered in the paper by Dzanelidze [Dzanelidze, 1958]. V. Z. Vlasov [Vlasov, 1959] has shown that this principle is not valid for thin-walled bars.

2.2.5. A notion of tension of flexible threads. G. Pardis, J. Hermann, Jacob Bernoulli. Studies in elasticity by L.E. Euler

It should be noted that, when studying the catenary and suspension bridge, G. Pardis (1673) supposed that the shape of a flexible line would not change, if its any part hardened, or the more, if we changed the thread parts distributed over the points $A$ and $a$, by the corresponding forces acting along the tangent line in points $A$ and $a$. This principle is used in all further studies of the catenary.

Jacob Bernoulli used the Pardis principle without changes in a number of his investigations of flexible threads of arbitrary thickness under the effect of distributed load (1691-1704). He introduces $firmitas$ (stress) in the explicit form; the latter being designated as $T$, we can record one of the formulas, in which Bernoulli gives general equations of equilibrium of a plane flexible thread as follows:

\[
T \frac{dx}{ds} = T_0 - \int_{0}^{s} F_x ds,
\]

\[
T \frac{dy}{ds} = -\int_{0}^{s} F_y ds,
\]
where \( F_x \) and \( F_y \) – components of applied forces per the length unit. These results were published before 1744; meanwhile, Jacob Hermann (1716), who was a follower of Jacob Bernoulli, developed them and published the explicit definition of the notion *tenacitas vel firmitas*: “The tension or compression of a thread or body at any of its points or at an element of the curve is that force of the thread or body which resists that power or force growing from all the applied powers [i.e. loads] and tending by pulling the thread in opposite direction to tear it apart. This tension exactly equal or is equipollent to that tearing force resulting from all the powers applied to the body”. This definition is not so informative as it looks at first sight, since Jacob Hermann did not consider something more general than a plane ideally flexible thread.

C. Truesdell [Truesdell, 1968] notes that Jacob Bernoulli and L. Euler in their works on bars bending (1705, 1727, 1779) studied the effect of material properties on the general notion of stress.

Jacob Bernoulli proved in his article of 1705 that the elasticity law for the bar fibers had to include fractional extension as a force function divided by the area. Maybe errors committed in other issues prevented his followers, even Antoine Parent (1666-1716), from using the ideas expounded in this article. Since the cross-section shape affects explicitly the modulus (as it was in Leibniz’s work about extensible and inextensible bars), no certain answer has been obtained by analysis. The first case, when it is mathematically necessary to use rather a general equation for material than the equation for various bodies, occurs, when we deduce Bernoulli’s bending law on the basis of the law of expansion for fibers to obtain the well-known current result of rigidity under bending \( EI \), where \( E \) – “Young modulus” of material, \( I \) – inertia moment of cross-section around the axis perpendicular to the axis bending area. The value \( I \) occurred in the above work by G. Leibnitz (1687) on essentially inextensible and inflexible bars. To integrate stress in the cross-section, as Leibnitz made it, it was necessary to have rather the expression for stress per area unit, than equipotent stress in the bar.
Though Jacob Bernoulli tried during many years to obtain the bending theory based on the law of fiber expansion, he never won a success, though had a lot of ideas and principles. It fell on the lot of L. Euler, who obtained a proof in attempting to study the elasticity theory, when he was a 20-year old student in Basel (1727); but this work remained unpublished till 1862. In this article he used the unidimensional tension-compression dependence similar to that offered 20 years ago by Jacob Bernoulli. But this dependence is linear, so, he had to introduce the value called now the Young modulus. L. Euler recurred to this question in 1779, at the end of his life. He presented and explained the definition of the “Young modulus” and similarity laws for bars based on its use in his last articles on bars vibrations published in 1782.

The vector character of stresses, though rather obvious, was not openly expressed in all these works.

As Truesdell [Truesdell, 1968] noted, T. Young himself did not, unfortunately, introduce this modulus, but recognized the law of elasticity which rather expressed the properties of material than of the body made of this material. All the fragments of T. Young’s lectures [Young, 1807], concerning elasticity and vibrations, consist of the simplest parts of Bernoulli’s and L. Euler’s researches, where equations are transferred to the language of words, and proofs are often insufficient.
\[
\begin{align*}
\frac{dT}{ds} - V\kappa &= -F_I \\
\frac{dV}{ds} - T\kappa &= -F_n \\
\frac{dM}{ds} - V &= 0
\end{align*}
\]

where \( \kappa \) – curvature, and \( M \) – bending moment. Later he expressed the principle

\[
S_+ + S_- = 0,
\]

where \( S = Tt + Vn \) and where signs plus and minus are responsible for the action of material removed on the both sides of the point under consideration.

2.2.6. S. Stevin’s principle of solidification. Works in hydrostatics, hydraulics and hydrodynamics

It is generally known that fluids exert pressure on the walls of vessels, containing them, but centuries of work of the human thought were required to develop a certain concept of internal pressure. The earliest mention of force of the fluid’s effect on fluid occurs in Archimedes’ theory (250 B.C.) about the spherical ocean covering the spherical Earth. The principle of solidification by Stevin is often formulated as follows: if any part of fluid is substituted by a rigid body, forces acting on the part of the rest of fluid remain unchanged. But such a formulation was not found up to the appearance of the work by A.C. Clairaut (1743).

Stevin’s theory (1586) is constrained by incompressible fluids on the surface of the plane Earth. S.Stevin obtains a correct formula for pressure on the horizontal bottom; despite the fact that he never made any certain statement concerning the lateral pressure, he succeeded in finding a total force on the surface slope, approaching them by stepped walls, and then passing to the boundary with 108
decreasing steps to zero. Stevin’s principle of solidification, giving several results, belongs only to the resultant force, which affects the plane bottom. In current designations it is a statement that regardless of the vessel form there is a ratio

\[ F = \rho ghA, \]  

where \( h \) – the depth of submersion of the plane bottom with the area \( A \) under the water surface.

Maybe hence follow the results of B. Pascal (1663) for incompressible fluids; it became known from Pascal’s book that air in the state of equilibrium follows the same laws as water.

Mathematical hydrostatics had been developed prior to the introduction of the concept of internal pressure. I. Newton (1687) was the first to distinguish mentally the internal part of fluid and to express a statement that it is “uniformly compressed on every side” and “any internal part of a fluid is in the same state with the submersed body.”

The actually used though non-formulated postulate in Taylor’s theory of atmosphere equilibrium (1715) is equivalent to the following ratio:

\[ dF = \rho ghdA, \]

where \( F \) – force, which would affect the plane surface of the area \( A \) at the height \( h \). Thus, Taylor thought that S. Stevin’s formula (2.9) might be applied to increments even in the fluid of variable density \( \rho \).

L. Euler in his treatise on hydrostatics (about 1740) considers only incompressible fluids near the Earth surface. He states in the explicit form that in all cases omnidirectional pressures are equal in a given point and normal to surfaces, which they affect. The work begins with a lemma, which is equivalent to the equation

\[ dF = \int_{S} \rho ghdS. \]
A.C. Clairaut and K. Maclaurin avoid using the notion of internal pressure in their works.

Daniel Bernoulli (1730) had successfully combined the velocity of a steady flow of incompressible fluid in a pipe with pressure of fluid taking effect on the walls.

The internal force affecting a thin fluid layer in the pipe was first introduced by Johann Bernoulli (1740) in the course of successful performance of his program of combining Daniel Bernoulli results in the framework of general system of mechanics. This pressure, which Johann Bernoulli called a motive force, is the action of fluid on fluid. Though they considered only movement of incompressible fluid in pipes, correct results for the pipes of arbitrary shape and for unsteady flows were obtained in this work. Really, Johann Bernoulli applied the notion of tension, developed by Jacob Bernoulli, when studying equilibrium of a flexible thread, to hydraulic motion.

L. Euler took up the idea of J. Bernoulli with enthusiasm and assumed it as a basis of his clear and general exposition of hydraulics in a number of works dedicated mainly to concrete problems (1749-1752).
He considered only incompressible fluids in these works. True hydraulic pressure $p$, having dimensionality [force][area], plays in these works the same role as in the current interpretation.

D’Alembert obtained in his works in hydrodynamics (1749) a number of important and rather general equations.

Papers by L. Euler in hydrodynamics (1750-1766) are based on using the internal hydrodynamic pressure in its all generality for any fluid in any conditions..

2.2.7. Bending of a beam

Having determined the absolute resistance of a bar, Gallilei studies fracture resistance of the same bar in case when it is used as a cantilever and loaded at a free end (Fig. 2.5,a). He states: “It is clear that, if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC, to which we the force is applied; the thickness of the solid BA is the other arm of the lever along which is located the resistance. This resistance opposes the separation of the part BD, lying outside the wall from that portion lying inside. From the preceding, it follows that the magnitude of the force applied at C bears to the magnitude of resistance, found in the thickness of the prism, i.e. in the attachment of the base BA to its contiguous parts, the same ratio which half the length BA bears to the length BC”\(^\text{15}\).

We see that in case of fracture in accordance with Gallilei’s presentation the “resistance” is distributed uniformly in the cross-section $BA$ (Fig. 2.5,b). We think that the bar cross-section is a rectangle and that material follows the Hooke’s law up to the fracture and obtain the stress distribution according to diagram shown in Fig. 2.5, c

\(^{15}\) Discorsi e dimostrazioni..., Dialogue two.
It should be noted that whatever be the stress distribution law under bending at the section \( \sigma = f(y) \) and wherever be a neutral line, the section modulus of the rectangular profile is \( w = \frac{bh^2}{m} \), and that of the circular one \( w = \frac{\pi d^3}{n} \), the numbers \( m \) and \( n \) being functionals of \( f(y) \), which do not depend on the cross-section dimensions. This theorem was first proved by A. Parent for the rectangular profile.

Several stress distribution laws were accepted in the future. Gallilei accepted \( f(y) = C \), E. Mariotte and H. Leibniz – the law \( f(y) = Cy \), considering the coordinate system origin at the cross-section margin, A. Parent accepted the same law but with the coordinate system origin in the middle of the height, and at last, L. Navier transferred the coordinate system origin (neutral line) to the cross-section gravity centre. Thereby the coefficients \( m \) and \( n \) for the rectangular and circular section had the value:

<table>
<thead>
<tr>
<th></th>
<th>Gallilei</th>
<th>Mariotte</th>
<th>Parent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>( n )</td>
<td>8</td>
<td>12.8</td>
<td>32</td>
</tr>
</tbody>
</table>

Note that the presented values may be considered as correct, only if the beam is in the elastic state. But beyond the limits of Hooke’s law the function \( f(y) \) changes its appearance, while Gallilei studied just the limit state of the beam. Thus, our ordinary values \( m = 6 \) and \( n = 32 \) are not valid for Gallilei’s problem. It is known that the modern theory of plasticity gives \( m = 4 \) and \( n = 18.84 \) in the limit state. The real values of these coefficients for the concrete material should be determined by experiments [Bernshtein, 1957].

Thus the problem on comparative strength of geometrically similar beams posed by Gallilei was correctly solved by him.

Two new fields of science created by Gallilei in his immortal work determined the development of dynamics and science of strength and remain urgent till now.
Based on his theory Gallilei obtains a number of important conclusions (Timoshenko, 1957). Considering a beam of rectangular cross-section he asks: “How and in what proportion doe a rod, or rather a prism whose width is greater than its thickness, offer more resistance to fracture when the force is applied in the direction of its breadth than in the direction of its thickness?”. Proceeding from this assumption (Fig. 2.5,a), he gives a correct answer: “Any given ruler or prism, whose width exceeds its thickness, offer more resistance to fracture when standing on edge than when lying flat, and this in the ratio of the width to the thickness”\(^\text{16}\).

In further studying of the problem on cantilever beam with constant cross-section Gallilei concludes that the bending moment of the beam weight increases in proportion to the square of its length. Preserving the length of circular cylinders, but changing radii of their bases, Gallilei finds that their resistance moment is proportional to radii cubes. This result follows from the fact, that the “absolute” resistance is proportional to the cylinder cross-section area, and the resistance moment arm is equal to the cylinder radius.

Comparing geometrically similar cantilevers, loaded by their own weight, Gallilei makes a conclusion that geometrically similar beams are not ones of equal strength.

All these reasons bring Gallilei to the following significant remark of general character: “You can plainly see that impossibility of increasing the size of structures to vast dimensions either in art or in nature; likewise the impossibility of building ships, palaces or temples of enormous size in such a way that their oars, yards, beams, iron-bolts, and, in short, all their other parts will hold together; nor can nature produce trees of extraordinary size because the branches would break down under their own weight; so also it would be impossible to build up the bony structures of men, horses, or other

\(^{16}\) See Discourses and Mathematical Proofs..., [Gallilei, 1934 (p. 228)]. A quotation presented by S.P. Timoshenko does not contain complete answer: there is only a reference to empirical fact; the theoretical explanation, based on the ratio of moments, is in the answer continuation. See, ibid. [Gallilei, 1934 (p. 229)].
animals so as to hold togethern and perform their normal functions if these animals were to be increased enormously in height; for this increase in height can accomplished only by employing a material which is harder and stronger than usual, or by enlarging the size of the bones, thus changing their shape until the form and appearance of the animal suggest a monstrosity... If the size of a body be diminished, the strength of that body is not diminished in the same proportion; indeed the smaller the body the greater its relative strength. Thus a small dog could probably carry on his back two or three dogs of his own size, but I believe that a horse could not carry even one of his own size” [Gallilei, 1934, (pp. 247-248)].

Gallilei also studies a beam resting on two supports (Fig. 2.6), and finds that the bending moment acquires the highest value in that point of the gap with applied load, which value is proportional to product $ab$, so, to make a fracture with the least load the load should be placed in the middle of the gap. He notes that there is a possibility of saving material, the cross-section near supports being decreased.

Gallilei presents a complete solution of the problem on the cantilever with equal support, which cross-section is a rectangle. Considering first the prismatic cantilever $ABCD$ (Fig. 2.7,a), he points out that some part of material may be removed from it, with no harm to its strength. He also shows that, if we remove a half of material and put a cantilever into the shape of a wedge $ABC$, the strength in any intermediate cross-section $EF$ will prove to be insufficient for the reason that, if the ratio of bending moments in the sections $EF$ and $AC$ is equal to the ratio $EC/AC$, the resistance moment for these cross-
sections is in the relation \((EC)^2/(AC)^2\). For the resistance moments were in the same ratio as the bending moments, we have to put the longitudinal cantilever outline into the parabolic shape \(BFC\) (Fig. 2.7,b). This meets the requirement of equal strength, since we have for the parabola:

\[
\frac{(EF)^2}{(AB)^2} = \frac{EC}{AC}.
\]

In conclusion Gallilei studies the strength of hollow beams, indicating that such beams “are emloped in art - and still more often in nature – in a thousand operations for the purpose of greatly increasing strength without adding to weight; examples of this are seen in the bones of birds and in many kinds of reeds which are light and highly resistant both to bending and breaking. For if a stem of straw which carries a head of wheat heavier than entire stalk were made up of the same amount of material in solid form, it would offer less resistance to bending and fracture. This is an experience which has been verified and confirmed in practice where it is found that a hollow lance or a tube of wood or metal is much stronger than wold be a solid one of the same length and weight...” Comparing a hollow cylinder with a solid one with the same cross-section area, Gallilei notes that their absolute rupture strengths are the same, and since the resistance moments are equal to absolute ones multiplied by the outer radius, the pipe bending strength exceeds the corresponding strength of the solid cylinder as many times as the pipe diameter exceeds the diameter of the solid cylinder.

It should be noticed that E. Mariotte, holding viewpoints of Gallilei as to the limit state of the bar, showed first that, when the bar bending fracture proceeds under rotation about the compressed rib (by Gallilei), forces in the menacing section would rather change following the linear law, than remain unchanged. After Mariotte the same triangular diagram was obtained by G Leibniz (1684) and P. Varignon (1702). Somewhat later E. Mariotte corrected this diagram and distributed a zero point in the middle of the height, recognizing
compression availability under bending and having approached the truth. In contemporary manuals the discovery of compression under bending is often ascribed to the French botanist and naval engineer Henri-Louis Duhamel du Monceau.

![Gottfried Wilhelm Leibniz](image)
![Pierre Varignon](image)
![Henri-Louis Duhamel du Monceau](image)

Gottfried Wilhelm Leibniz (1646–1716)  
Pierre Varignon (1654–1722)  
Henri-Louis Duhamel du Monceau (1700–1782)

There occurred an annoying case. When calculating resistance of the bar with rectangular bending cross-section using a two-valued stress diagram under the condition of equality of external and internal force moments, E. Mariotte made an algebraic error and introduced a redundant two in calculation. As a result he came to a conclusion that the beam strength was the same under the triangular diagram and two-valued stress diagram with zero in the middle of the height. In both cases he obtained the values of resistance moments of rectangular cross-section $(bh^2)/3$ instead of $(bh^2)/6$ for two-valued diagram.

Strange as it may seem but the same error occurred in calculations of other well-known scientist 25 years after E. Mariotte. That caused a 100-year delay (!) of the correct solution of the problem. In 1705 Jacob Bernoulli recurred to the bending problem after he had established the law on proportionality of a bending moment of beam curvature. Like E. Mariotte, J. Bernoulli compared a triangular diagram and two-valued stress diagram and also made a double mistake at the same place, when estimating strength by the same diagram. Based on that Bernoulli even formulated a strange
theorem that the position of zero point in a linear stress diagram does not affect in any way the bending resistance, and thus, it is not worth specifying this point position. Just this improper theorem confirmed by J. Bernoulli’s authority had hampered the study of bending for the whole century.

A. Parent, a French military engineer, was the first to make a proper solution of the problem on beam bending strength in 1713 [Parent, 1713]. He has shown a necessity of equilibrium between the tension and compression forces under bending, hence the zero point of the diagram is in the middle of height under the symmetric section of the beam. Besides, Parent was the author of solution of the known problem how one can cut a rectangular bar with the highest bending strength of a circular billet.

Antoine Parent, a French mathematician and mechanist, was a pupil of P. de La Hire. He was also engaged in anatomy, botany, chemistry, etc. His works are dedicated to analytical geometry and spherical trigonometry. He developed the beam bending theory, corrected some E. Mariotte’s results.

A correct solution of A. Parent, who obtained the expression of resistance moment for a rectangular section \((bh^2)/6\), was repeated by Petersbourg Academician G. Bilfinger. His work [Bilfinger, 1735], published in Latin in Vedomosti Akademii nauk (Records of the Academy of Sciences) in 1735, was the first work in structural mechanics published in Russia.

Ch. A. Coulomb repeated A Parent’s conclusion 60 years later. It was important that Ch. A. Coulomb brought the position of the neutral line in consistency with Hooke’s law. He indicated, in particular, that in case when Hooke’s law ceases to act before the fracture, the neutral line will take other position under the fracture. This Ch. Coulomb’s statement was the first that indicated essential divergence between the working and limiting states.

Thus, in 1773 Ch. A. Coulomb completely confirmed correctness of A. Parent’s work of 1713. In his famous memoir (1773) Ch. A. Coulomb reviewed and justified A. Parent’s arguments. He
considered a case of arbitrary change of both normal and tangent forces affecting the transversal section of the beam under transversal loading applied to its ends. As it was with A. Parent, these forces may be considered as internal stresses in a plane case. Ch. A. Coulomb has really recorded in integrated form all the equilibrium equations for the transversal section. Thus, he came to a conclusion that the area bounded by the pressure curves had to equal the area bounded by the tension curve, and he calculated resulting bending stresses depending on loading. Ch. A. Coulomb was the first after A. Parent, who considered stresses affecting various planes, crossing the point; he calculated the bending stress on the area bent arbitrarily to the direction of the applied force for a beam, which, by assumption, is in the state of pure compression. He also showed that the displacement is the biggest in the area, sloped at the angle of 45°. But, as it was noticed by B. Saint-Venant, the work by Ch. O. Coulomb included such a great quantity of ideas in a small volume, that the attention of engineers and researchers could not dwell on any of them during the following half-century.

P. Girard’s Course of Material Resistance was published in 1798, 25 years after Ch. A. Coulomb’s work. P. Girard refers to Ch. A. Coulomb in solving the bending problem, but, despite this, places the diagram zero point at the section edge and repeats the improper statement of Jacob Bernoulli that the zero point position may be arbitrary. Strangely enough, but this book received a positive review of Ch. A. Coulomb himself. That is P. Girard returned the bending problem to its level of 1713 (a 85-year regress).

Fifteen years more had passed, and L Navier in one of his first works (1813) just kept to the theorem of Bernoulli (that is 100 years after A. Parent had proved its erroneousness). This may only be explained by a perfunctory speculative treatment of the bending problem and by the lack of experiments and their theoretical justification, respectively.

In 1820, A. Dulot made a statement that the position of the zero point in the stress diagram is determined issuing rather from the
condition of equality of the moments of tensile and compressive stresses than the equality of forces themselves. If we consider that
\( \sigma = ky \), this Dulot’s condition looks as follows
\[
\int_{p} \sigma y dF = \int_{c} \sigma y dF ,
\]
where indices \( p \) and \( c \) under the integral signs indicate that integration occurs in the tensile and compressive zones of the beam section, or
\[
\int_{p} y^2 dF = \int_{c} y^2 dF .
\]

That is the inertia moments of these two section zones are equal. Evidently, it is true only for the symmetric profile of a beam.

T. Tredgold [Tredgold, 1821] had come to the same result. He offered a term neutral line, and treated it as follows: “a neutral line divides the section into such two parts that, each of them being symmetrically reflected on the other side of the neutral line, the both symmetrical profiles will show the same resistance” This rule is practically equivalent to A. Dulot’s conclusion. But the question is that rigidities of the both profiles, constructed in such away, will be the same, while strength is different. The Dulot-Tredgold idea was probably the last mistake in the history of the beam bending problem. In 1826 L. Navier presented at last a true problem solving, though he admired (not for a long time) Dulot’s theory.

N. Persy [Persy, 1834] made the next step after L. Navier in this field. He introduced a concept of section inertia moment, developed the theory of inertia moments and indicated that the Navier bending theory is true only if the neutral line coincides with the principal section axis. Nine years later B. Saint-Venant solved the problem of oblique bending (1843), when introduced two equations of moments in respect of the both principal axes. He also solved the problems on bending with tension having used for the first time a principle of superposition of small deformations.
It is a striking fact that it had taken 20 years since appearance of first elasticity theory equations before their linearity led to the thought on possibility of such superposition. It is even a more striking fact that a thought of possible superposition of loads appeared much later, by the end of the 1840’s, when it was simultaneously used by D.I. Zhuravsky and G. Bresse. Remember, that the linearity of elasticity theory equations was a result of assuming the condition of small deformations and unbounded validity of Hooke’s law, i.e. those admissions, which became the essence of revolution in structural mechanics – the transition from the principle of limit state to initial operative condition analysis. Additional advantage of this transition is linearity of all equations, and just this was not comprehended and used during two decades.

The problem of the eccentric tension or compression was solved in general form by G. Bresse (Bresse, 1854), who had introduced the notion of the core of a cross-section and studied its properties.

2.2.8. Determination of shear stress. Coulomb

The well-known memoir by Ch. A. Coulomb [Coulomb, 1773] contains correct solutions of a number of problems of material mechanics, but it took more than 40 years before engineers could completely comprehend and use them in practice.

Charles-Augustin de Coulomb (1736-1806) was born in Angulem (France). After training in Paris he entered the military-engineering corps.
Being detached to Martinica Island, he took part there in different construction works that made him to learn mechanical properties of materials and other problems of building engineering. Staying on the island he wrote a well-known work *Sur line application des regies de maximis et minimis a quelques problemes de statique relatifs a l'architecture* ("On Application of Minimum and Maximum Rules to Some Problems of Statics Relating to Architecture"), presented in 1773 to French Academy of Sciences. In this work preface Coulomb reports: "This paper, written several years ago, was originally mean only for my own use, in the different tasks in which I was engaged in my profession; if I dare to present it to this Academy, it is because the weakest work is always received kindly by it if the subject is of practical use. Moreover, the Sciences are memorials dedicated to the public good; each citizen should contribute to them according to his capabilities. While great men, installed in the roof of the building, design and build the upper storeys, ordinary workmen, scattered in the lower storeys, or hidden in the darkness of the foundations, should try only to perfect that which more capable hands have created."

After 1781 Ch. A. Coulomb resided in Paris, where he was elected member of the Academy and was given ample opportunities for scientific work. He went in the new field of research – electricity and magnetism. He had invented a rather sensitive torsional balance for measuring low electrical and magnetic forces, and in this connection studied torsional strength of the wire.

In the beginning of French revolution of 1789 Ch.A. Coulomb left for his small estate in Bloi. In 1793 the Academy was closed, but two years later it started its work, being renamed into Institut national des Sciences et de Arts (National Institute of Science and Arts). Ch.A. Coulomb was one of the first elected to this new educational institution, and his last works dedicated to the problems of fluid
viscosity and magnetism were published in Memoires de l’Institut (1801, 1806). In 1802 he was appointed to the post of one of general inspectors in science and made great efforts to improve organization of national education. This activity was connected with frequent trips, which were too exhausting at his age and poor health; he died in 1806. But his works are still significant, and we use his theories of friction, strength of building materials and torsion.

No other scientist of the 18\textsuperscript{th} century contributed so much to mechanics of elastic body as Ch. A Coulomb [Timoshenko, 1957]. His most valuable achievements in this field entered into his work published in 1773. There he wrote about his tests in determining strength of one of the kinds of sandstone.

In his theory of bending Ch. A. Coulomb properly used the equations of statics, when studying internal forces, and understood clearly the distribution of these forces in the beam cross-section. Coulomb was not supposedly acquainted with A. Parent’s works, since he refers only to Bossu, who recommends in his work La construction la plus avantageuse des digues (Most Profitable Construction of Dambs) to calculate woody beams as elastic, and stone beams as absolutely rigid ones.

Coulomb saw his next task in studying prism compression by the axial force $P$. Coulomb had first mentioned the criterion of plasticity (1733), known in the courses of material resistance as the criterion of the highest tangential stress. He thought that the fracture of the compressed prism appears as a result of one part sliding against another at a certain plane distributed at the angle of 45° to compressing force. But Coulomb groundlessly thought that the tangential stress at this plane (maximum) under sliding is rather equal to the limit stress under tension than to its half. Thus, though Coulomb had taken the value of maximum tangential stress as the criterion, but he rather used it as the fracture criterion.

In 1784 Coulomb published his memoir on torsion. Coulomb has certainly played a progressive part in the history of structural
mechanics. He has discovered a wonderful method for calculation of retaining walls, he also has works in arches design.

We have already noted that Coulomb gave correct solution of the beam strength problem. Besides, he has made three important discoveries. Firstly, he has found the existence of tangential stresses. This discovery is probably connected with his well-known researches in the field of friction. Coulomb showed that the fracture of compressed bars often proceeds by shear and made a supposition that just shears cause various fractures. Now we call this supposition the third theory of strength and use it widely in the design practice.

Interestingly that Coulomb had given the problem on tangential stress in a compressed bar a form of the minimum problem, as well as all other problems of structural mechanics considered in his work, which was titled: Application of the Rules for Finding Minimums and Maximums to Some Statics Problems Related to Architecture. That is the true text of this problem: “To determine the highest weight that a base can bear, one should find a section in which the cohesion forces can balance the lowest weight”. This minimum weight for a dangerous cross-section will be maximum safe load weight for the base as a whole. The same original approach – determination of maximum loading as minimum of a certain function – Ch.A. Coulomb also applied to calculation of retaining walls and arches. That was a sequential way of the limit-state analysis.

Secondly, Coulomb was the first to solve the torsion problem (1784). He had studied the torsion of thin wires of a circular section and deduced true calculation formulas, which we always use without knowledge of their author. Coulomb had established that the angular moment was proportional to the torsion angle and to the fourth degree of the diameter. Now we would say that the torsion angle is proportional to torsion moment. But for Coulomb the design state was the final state of the bar, which had already been strained. From this strain Coulomb determined a torsion moment which caused it. Coulomb’s precursors treated the beam deformation problem in the
same way: they determined load from the set deformation which caused it. The approach was typical of the 18th century.

Coulomb was also the first, who had experimentally established a formula for the torsion angle of the circular bar. In memoirs presented to the Paris Academy of Sciences in 1784 [Coulomb, 1784] Coulomb gave results of experimental study of torsional vibrations of a circular bar, obtained differential equation of free torsional vibrations and deduced a formula for the period of vibrations. When studying torsional vibrations in experiments with the use of a special device, Coulomb established that the period does not depend on the torsion angle under small torsion angles. On this basis Coulomb made a correct conclusion that the torsion angle is proportional to the torsional moment.

Coulomb tested circular wires of various length and diameter for torsion, he obtained a formula in the experiment for the torsion angle of a circular bar $l$ long:

$$\phi = \frac{kM_t l}{d^4},$$

where $M_t$ is a torsional moment, $k$ – a constant for material. Now we know that

$$\phi = \frac{M_t l}{G I_p},$$

where $G$ – modulus of shearing. Thus, Coulomb introduces the modulus of shearing and determined it experimentally for iron and brass.

Thirdly, proceeding from experiments for bending and torsion Coulomb deduced a law of proportionality between the load and elastic part of deformation up to fracture – the law which is called erroneously the Herstner law.

Coulomb’s brilliant discoveries were the last important researches in the field of structural mechanics of the 18th century.
2.2.9. Allowance for shear under bending of beams. D.I. Zhuravsky

Shear availability under bending of a beam was first indicated by H.L. Duhamel du Monceau in 1767, who had noted that the distinction of strength and rigidity in a continuous beam and in a beam composed of separate layers may be only explained by shears. Later Ch. O. Coulomb proved that Duhamel’s calculation of bending is true only in the cases when the force arm is much more than the section height (or in modern terms, if the effect of the cross force on strength is low). Then the notion of shear under bending occurs in Young, who compares this phenomenon with action of scissors. The term of cross force was introduce by L. Vicat, who studied experimentally the shear under bending [Vicat, 1833]. Thus the phenomenon of shear under lateral bending had been known since D.I. Zhuravsky tried to calculate it and gave his well-known solution [Zhuravsky, 1855].

The problem on tangential bending stress was a side problem for D.I. Zhuravsky; it arose when solving a more general one on design of woody girders.

Note, that structural mechanics developed rather positively in Russia of that time. In 1810 the first higher technical educational institution was opened in Russia – Institute of Railway Engineers in Petersbourg that became a cradle of Russian engineering sciences. For the lack of home specialists its creation was committed to the most progressive scientific schools of that time – the French one. That was a good choice, since neither German nor English schools could compete with the French school. Some well-known French engineers – Betancourt, Bazaine, et al., were invited then to Russia; in the 1820’s we meet there the names of such outstanding scientists as G. Lame and B. Clapeyron.

Thus, it took almost two centuries, to be more precise 188 years, to solve the beam bending problem, when counting from the date of publication of Gallilei’s Dialogues, where he first tried to calculate the beam strength. That is a typical example of that thorny path, which
science followed in the 18\textsuperscript{th} century. By the way, the notions of inertia moment and section resistance moment were not separated during all that century.

In 1848 D.I. Zhuravsky developed an approximate method for beams calculation (published in 1855-1856).

D.I. Zhuravsky was a practical engineer, bridge builder, and he remained a convinced practitioner in his scientific activities as well. His works contain no theoretical generalizations, which could be of use in the future. He proceeded from the construction requirements and tried to meet them with a wonderful courage, not being afraid of any difficulties. His scientific intuition and resourcefulness were striking, and just these qualities allowed him to solve the problem on shear under bending. To evaluate his deserts in this field we should remember that it had taken 200 years of work of the greatest scientists from Gallilei to Navier before a true decision was obtained for a simpler problem of normal bending stress. D.I. Zhuravsky had posed and solved himself the problem of shear, with not a hint on the method of solving in the works of his predecessors. D.I. Zhuravsky considered one partial problem: a cantilever of rectangular profile with load at its end (Fig. 2.8). If the cantilever beam is detached from restraint and normal forces distributed by two-valued curves in the reference section are applied instead of support moment, it becomes clear that the omnidirected normal forces cause a cut along the
neutral layer. The value of shearing force $Q$ is determined from equilibrium equation

$$Q = \frac{h^2}{2} \int_0^h \sigma dF = \frac{M}{J} \int_0^h y dF = \frac{MS}{J}.$$  

At first sight this solution may seem strange and even incorrect, since a shearing force proves proportional to a bending moment. But this doubt is easily explained; a moment at the free end of the cantilever beam is equal to zero, but if it differed from zero, the shearing force would be proportional to the difference of moments, or, to be more precise – a moment derivative. As is known, the last conclusion is often called the theorem of D.I. Zhuravsky, though he has not given its direct proof.

A shearing force and expression for tangential stresses in the neutral layer may be obtained from the formula, to do this it should be set that stresses are uniformly distributed on the width of cantilever $b$ and its length $l$

$$\max \tau = \frac{Q}{bl} = \frac{3}{2} \cdot \frac{P}{F}.$$  

D.I. Zhuravsky also gave a more general solution for tangential stresses in arbitrary layer, which result after dividing by $b$ and $l$ in the known formula:

$$\max \tau = \frac{6P}{bh^3} \left( \frac{h^2}{4} - y^2 \right) = \frac{PS}{Jb}.$$  

However it should be noticed that the method of solution, based on the admission on uniform distribution of tangential stresses on the beam length, is valid only for the beam section, which bears no local
load. D. I. Zhuravsky considered just this case, but has not stipulated this.

In foreign literature the formula of tangential stresses derived by
D.I. Zhuravsky first occurs in W. Rankine [Rankine, 1862] in 1862. It is known that formula of D.I. Zhuravsky is approximate, since it requires the adoption of hypothesis on uniform distribution of tangential stresses on the width of rectangular profile. An exact solution obtained with time by Saint-Venant by the methods of elasticity theory, has shown high degree of approximation of Zhuravsky’s formula for rectangular profiles; their height being equal or exceeding the width, i.e. for practically all important cases. Now we also freely use D.I. Zhuravsky’s formula for more complex forms of profiles – double-T shape, structural channel, etc., since this solution proved considerably more general, than D.I.Zhuravsky supposed it himself. And it is especially significant that Zhuravsky’s solution remained the only practical method of design of tangential bending stresses.

Owing to results of the work by D.I.Zhuravsky, who had solved the problem of tangential bending stresses, and activities of the French school, which gave solution for normal stresses under any shape of the profile, the bending problem for rectilinear bar proved to be rather completely solved by the mid-1850’s.

References


Bernoulli, D. (1950), Gidrodinamika [Hydrodynamics], Izdatelstvo AN SSSR, Leningrad, USSR.

Bernoulli, J. (1937), Izbrannye sochineniya po mekhanike [Selected works in mechanics]. Moscow-Leningrad, Russia.

17 There exists translation into Russian: [Rankine, 1870]
Bershtein, S.A. (1957), *Ocherki po istorii stroitelnoi mekhaniki* [Essays in the history of structural mechanics], Gosstroyizdat, Moscow, Russia.

Vlasov, V.Z. (1959), *Tonkostennyie uprugie stershni* [Thin-walled elastic bars], Fizmatgiz, Moscow, Russia.

Galileo Galilei (1934), *Sochineniya Tom. 1,* “Besedy i matematicheskie dokazatelstva, kasayushchiesya dvukh novykh otрасlei nauki, otnosyashchikhsya k mekanike i mestnomu dvizheniyu” [Collected works, Vol. 1. Dialogues and mathematical proofs concerning two new fields of science, related to mechanics and local motion], Ed by S.N. Dolgov, Series “Classics of Natural Science”, Gosudarstvennoye tekh.-teor. Izdatelstvo, Moscow-Leningrad, USSR.

Descartes, R. (1938), *Geometriya. S prilozheniem izbrannykh rabot P. Ferma i perepiski Dekarta* [Geometry. With supplement of selected works by P. Fermat and correspondence of Deacartes], GONTI, Moscow-Leningras, USSR.

Descartes, R. (1989), *Sochineniya v dvukh tomakh* [Collected works in two volumes], Ripol Klassik, Moscow, USSR.


Lyav, A. (1935), *Matematicheskaya teoriya uprugosti* [Mathematical Theory of Elasticity], Moscow-Leningrad, ONTI NKTP USSR.


Timoshenko S.P. (1957), *Istoriya nauki o soprotivlenii materialov s kratkimi svedeniymy iz istorii teorii uprugosti i teorii sooruzheniy* [The history of science of strength of materials with brief data from the history of elasticity theory and theory of structures], Gosteorizdat, Moscow, USSR.


Truesdell, K. (2002), *Ocherki po istorii mekhaniki* [Essays in the history of mechanics], Institut kompyuternyh issledovaniy, Moscow-Izhevsk, Russia.


Euler, L. (1959), *Dissertatsia o printsipe naimenshego deistviya s razborom vozrazheniy slavneishego prof. Koniga, vydvinitvykh protiv etogo printsipa. Variatsionnye printsipy mekhaniki* [Thesis on the principle of the least action with consideration of objections of the greatest prof. König advanced against this principle. Variational principles of mechanics], Fizmatgiz, Moscow, Russia.

Euler, L. (2002), *Pisma k nemetskoi printsesse o raznykh fizicheskih i filosofskikh materiyalakh* [Letters to German Princess on various physical and philosophical matters], Nauka, St.Petersbourg, Russia.


Bernoulli, J. *Opera omnia tam antea sparsim edita quam haec tum inedita: tomos primus, quo continetur ea quae ab anno 1690 ad annum 1713 prodierunt*, Lutetiae Parisiorum: apud Carolum Jombert.


Boscovich, R.I. (1763), *Theoria Philosophiae Naturallis redacta ad unicum legem virtium in natura existentium*, Venice, Italy.


Cauchy, A.-L. (1827a), De la pression ou tension dans un corps solide. *Exercices de Mathématiques*. 2, pp. 42–56


Coulomb, C.A. (1773), Essai sur une application des règles de maximis et minimis à quelques problèmes de statique, relatifs à l'architecture, Mémoires de mathématique et physique présenté à l'Académie des sciences par savantes étrangères, Paris, France.

Coulomb, C.A. (1776), Le statique des voûtes, Paris, France.


Culmann, K. (1866), Die graphische statik.

D'Alembert (1743), Traité de dynamique, dans lequel les lois de l'équilibre & du mouvement des corps sont réduites au plus petit nombre possible, David L'aîné, Paris, France.

Delambre, J.B.J. (1867), Notice sur la vie et les ouvrages de Lagrange Œuvres de Lagrange, V.1), Paris, France.

Duham, P. (1905), Les Origines de la statique, Paris, France.


Euler, L. (1749), Scientia Navalis seu Tractatus de Construendis ac Dirigendis Navibus, Typis Academiae Scientiarum, St. Petersburg, Russia.


Fahie, J.J. (1903), Galileo, his life and work, New York, USA.


Galileo Galilei (1638), *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, Leida.


Hermann, J. (1716), *Phronomia, sive de viribus et motibus corporum solidorum et fluidorum libri duo*, Amsterdam, Netherlands.


Mariotte, E. (1717), *Euvres de Mr. Mariotte*, Leyden, Holland.


Newton, I. (1687), *Philosophiae Naturalis Principia Mathematica*.


Persy, N. (1834), *Cours de stabilité des constructions a l'usage des eleves de l'Ecole d'application de l'artillerie et du genie*, Metz, France.


Stevin, S. (1586), *De Beghinselen der Weeghconst*.


Thomson, W. and Tait, P.G. (1867), *Treatise of natural philosophy*.


Truesdell, C. (1966), *Six lectures on modern natural philosophy*, Springer-Verlag, New York, USA.


Varignon, P. (1687), *Projet d'une nouvelle mécanique*.

Varignon, P. (1704), De la resistance des solides en general pour tout ce qu'on peut faire d'hypothesis touchant la force ou la tenacite des fibres des corps a rompre; et en particulier pour les hypotheses de Galilee et de M. Mariotte. *Histoire de l'Academie Royale des Sciences Annee* 1702, Paris, France.


Vicat, L. (1833), Memoires et recherches experimentales sur les phenomenes physiques precedent et accompagnement la rapture ou l'affaissement d'une certaine classe de solides, *Annales des ponts et chaussées*, 2 semestre, 1833, pp. 201-268.

Vicat L. (1834), Note sur l'allongment progressif du fil de fer coumis a diverses tensions. *Annales des ponts et chaussées*, 1 semestre, 1834, pp. 40-44.


Essay 3

STAGES OF DEVELOPMENT OF STATIC STABILITY THEORY
The concept of stability is of fundamental importance. Only stable phenomena and processes may be used for more or less long time both in nature and in human active work.

V.V. Bolotin

One should repair a roof before repairing a floor.

Internet aphorisms
 Thousands of works are dedicated to the problem of static stability, and their historical analysis is an extremely complex task. But we can limit the field of research, dwelling mainly on the bar systems, which remained the only subject of research in structural mechanics for a long period of time. Such a decision is more relevant, since almost all significant results were obtained just in the process of analysis of stability of the bars and bar systems. As to the boundless field of stability of plates and shells, we will touch it only in that part, where the finite-element methods are used as a research tool.

3.1. General principles and theorems of stability

The notion stability, beginning from old times and to the 17th century, was connected with the problem of choice of true (observable) equilibrium states. Such a problem was probably posed in Mechanical Problems assigned to Aristotle. In so doing a problem was considered whether a body, being out of equilibrium, will return to initial configuration after the disturbance elimination. This question was raised concerning T-like balance, and Aristotle indicates that in case of upper suspension (Fig. 3.1, a) such a return takes place (stable position), and in case of a lower suspension (Fig 3.1, b) it will not happen.

 Archimedes was concerned with the same problem; he considered the state of equilibrium of a heavy parabolic segment,
which floated in fluid of different density; being a real practician, he was only interested in a stable equilibrium position, and he tested whether a body, disturbed from an equilibrium, can return to it itself.

Archimedes, Heron of Alexandria and other ancient thinkers considered this problem; Leonardo daVinci, Tartalia, Cardano, Stevin and other scientists of Renascence were also interested in it [Moiseyev, 1949]. They were interested in general criterion which could help determine without calculations, whether this position of equilibrium was stable or not. The first such criterion for mechanical system of bodies under the action of gravity force was defined by Torricelli, a follower of Galileo; he indicated (1664) that under the action of the forces of gravity the stable equilibrium corresponds to such system positions, in which the weight center height is of minimum value. For a common case, when a heavy body has only one point of support, Torricelli’s principle was interpreted as follows: the equilibrium position is stable, when the gravity center is below the point of support.

At new times we should reckon from the work Ship-Building Science by Leonard Euler. That was the first monograph in the theory of stability [Euler, 1749]. This treatise by Euler was caused by needs in ship-building practice, it laid the foundations of two most important sections of the theory of stability: theory of statical stability and theory of small vibrations near the equilibrium position.

A distinct definition was very important: “...the equilibrium position of a body is stable, if this body, being inclined, straightens again”, as well as the proposition as to quantitative characteristic of stability: “Stability of equilibrium position of the body floating in water is to be estimated by the value of the moment of restoring force, when the body is inclined from the equilibrium position by a certain infinitely small angle”.

Leonhard Euler
(1707-1783)
This stability notion for rigid bodies was extended to elastic bodies. Equilibrium of the elastic system is considered stable by Euler at preset external forces, if after static application and further removal of a small perturbing force the system returns to its initial state.

The first volume of *Analytical Mechanics* by Lagrange, which appeared in 1788 [Lagrange, 1788], included paragraph V, which dealt with “equilibrium properties related to maximum and minimum”. There was the following theorem:

*If at the equilibrium state under study the system potential energy takes strictly minimum value in a certain vicinity of this state, this equilibrium state of a conservative mechanical system is stable in terms of the above definition.*

As this takes place that is very important, the notion of equilibrium stability is connected with those movements, which can arise after disequilibrium, i.e., if the system in its further motion will slightly deviate from the equilibrium position under study, such an equilibrium state is considered stable.

A case of equilibrium stability of the system of bodies, which has several degrees of freedom, was considered by Lagrange by means of the method of “first approximation”, when it was reduced to consideration of the system of linear differential equations; if equations corresponding to harmonic vibrations were obtained for small deviations, the equilibrium position was considered as stable one. To do this it is necessary that the roots of characteristic equation, which helps determine possible periods of vibrations, were imaginary. Besides, Lagrange illustrated his theorem, resorting to Torricelli’s principle, applied to imaginary mechanism, in which forces were formed by tension of a thread transferred through tackles (Fig. 3.2).
Lagrange had not justified the rejection of the terms of higher orders, and the first strict proof of this theorem, based on the law of energy conservation and not attached to series expansion of potential functions, was given by Lejeune Dirichlet [Lejeune Dirichlet, 1846]. So there is the Lagrange-Dirichlet theorem.

![Fig. 3.2. Load transmission system by Lagrange](image)

Note, that Euler’s idea of equilibrium stability rests on absolutely different definition, where the notion of movement is not used at all. In his pattern of arguments the value of external load is mentally accumulated, and it is foreseen that in the process of such accumulation at any moment jointly with the given equilibrium state there arises a possibility of realization of the other equilibrium state allied with the studied one (the considered equilibrium state of the system ceases to be unambiguous). The load parameter value $\lambda = \lambda_c$, under which such equilibrium form ambiguity is admitted by mechanical system, is called critical (by Euler). It is also considered that, when the load values are less than Euler’s critical value, the equilibrium state is stable, and when the load values are higher than Euler’s critical value $\lambda_c$, the equilibrium state in unstable. It appeared that under certain conditions the both approaches lead to similar results, but the fact of their identity was comprehended later [Moiseyev, 1949].
The next principally important step in development of general principles of stability analysis was made by Henri Poincare, a well-known French mathematician, physicist, astronomer and philosopher, who studied the nature of stability loss of mechanical systems in parameters of generalized coordinates. The Poincare theory describes well the nature of stability loss of continual structures with the change of one parameter of the external load.

This theory allowed finding the bifurcation points (Fig. 3.3,a) and limit points (Fig. 3.3,b) on the curve of equilibrium states, the exact definition of the limit points belongs to Poincare [Poincare, 1882].

To evaluate the equilibrium quality Poincare considers a function of potential energy of the system, which he presents in a form of quadratic function of generalized coordinates \( q_i \)

\[
U = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} q_i q_j ,
\]

which in the stable equilibrium state should be additionally studied (that guarantees a strict minimum, which is required by Lagrange)

Fig. 3.3. Bifurcation points and limit points
Poincare determines positive definiteness of the function of the system potential energy by analysis of signs at squares of variables, when this function is reduced to canonical form

$$U = \sum_{j=1}^{n} a_j q_j^2 .$$  \hspace{1cm} (3.2)

Poincare has called coefficient $a_j$ at quadratic variables the series of stability, and the number of negative coefficients – the step of instability [Poincare, 1885].

The development of general principles of stability for conservative systems with the finite number of the degrees of freedom was completed by the works by A.M. Lyapunov. A problem of motion stability (in a separate case – equilibrium) [Lyapunov, 1892] was thoroughly considered in his thesis for doctor’s degree General Problem on Motion Stability.

Lyapunov offered an original, rather important method based on consideration of quasi-energy function, which bears his name now. Lyapunov opened the content of his method in general theorems on stability and instability.

Using this method, he has strictly revealed, when the problem on stability in the first approximation can be solved and when it cannot be solved. Lyapunov’s function is a scalar one:

$$V(X) = V(x_1, \dot{x}_1, x_2, \dot{x}_2, ..., x_n, \dot{x}_n),$$  \hspace{1cm} (3.3)

set in the phase space of displacements $x_j$ and velocities $\dot{x}_i$ of the system, using which one can prove the equilibrium position stability. Its complete time derivative may be recorded in a form of scalar product of two vectors

$$\frac{dV}{dt} = \left( gradV, \frac{dX}{dt} \right).$$  \hspace{1cm} (3.4)
Here, the first vector is a gradient of the function $V(X)$, i.e. it is always directed towards the greatest increase of the function $V(X)$. The other vector in the scalar product is the vector of motion velocity. In any point it is directed along the tangent line to the phase trajectory.

Consider a case, when a derivative of the function $V(X)$ in the vicinity $U$ of coordinates origin is negative:

$$\frac{dV}{dt} = \left(\text{grad} V, \frac{dX}{dt}\right) < 0.$$ \hspace{1cm} (3.5)

That means that the angle $\phi$ between the gradient vector and velocity vector is above 90°. This is schematically shown in Fig. 3.4 for the function of two variables.

Evidently, if a derivative $dV/dt$ along the phase trajectory is negative everywhere, the motion trajectory goes to coordinate origin, i.e. the system is stable. In the other case, when a derivative $dV/dt$ is positive, the trajectory is directed from the coordinate origin, i.e. the system is unstable.

![Fig. 3.4. Lyapunov’s function and its time derivative](image)

Thus, Lyapunov’s functions permit establishing the system stability and instability. This method merit is that one has not to know $X(t)$ solution itself. Besides, the given method allows studying the equilibrium positions of nonrough systems, for example, when
the equilibrium point is a center. A demerit is that there is no a
general method for constructing Lyapunov’s functions. In a specific
case of homogeneous autonomous systems with constant coefficients
the Lyapunov function may be searched as a quadratic form.

The last fact gives grounds to use a
functional of full potential energy of the
system as Lyapunov’s function, when solving
linearized problems. It was probably G. Bryan
[Bryan, 1891], who was the first to use the
energy approach (even without mentioning
Lyapunov’s theorems) for the continual system
of such type. He was the first to apply the
energy principle to solution of the problems on
equilibrium stability; proceeding from this
principle he obtained a differential equation
assumed as the basis of the theory of buckling of plane plates.

Studying stability of the infinite plate under the compression
action along the edges and setting deflexions by trigonometric series,
Bryan equates the work of compressing force to potential bending energy

\[
\frac{1}{2} P \int_0^L \left( \frac{dw}{dx} \right)^2 dx = \frac{EI}{2} \int_0^L \left( \frac{d^2w}{dx^2} \right)^2 dx
\]

(3.6)

and obtained the value of critical force.

In his work [Bryan, 1888] Bryan has found out the limits of
application of Kirchhoff theorem concerning the unity of solution of
elasticity theory problem and has shown that it does not function
under small deformations, if only one or two body sizes may be
considered small. Therewith the phenomenon of instability may take
place within the limits of elasticity, if the product of elasticity
modulus \( E \) by the square of the ratio of small dimension to infinite
one will be of the same order as the material elasticity limit.
Further development of the general theory of equilibrium stability of elastic bodies belongs to R. Southwell [Southwell, 1913]. He removes constraints as to deformation smallness and operates with the ideal body of infinitely big strength.

Following Bryan the energy approach was further developed in the works by S.P. Timoshenko; beginning from his first work of 1907 [Timoshenko, 1907], where the energy approach was used, he applied it widely to most various problems of stability of elastic systems [Timoshenko, 1910], including energy deduction of Euler’s formula [Timoshenko, 1910]. In contrast to G. Bryan, who used the elastic potential, which depends on the stressed state of a body, S.P. Timoshenko considered the work of internal forces on dislocations that correspond to the form of stability loss. That excluded a necessity of determining the initial stressed state. That is why that instead of (3.6) Timoshenko uses

\[
\frac{1}{2} P \int_0^l \left( \frac{dw}{dx} \right)^2 dx = \frac{P^2}{2EI} \int_0^l w^2 dx
\]

(3.7)

S.P. Timoshenko, like G. Bryan, as well as many other successors of the energy approach, used the variant of strongly linearized posing of the problem, when the system initial state, which equilibrium stability is studied, is considered as stressed, but not strained one.

It is just strong linearization in posing the problems of equilibrium stability that is traditionally (though it is not always justified) used in the applied theory of elasticity, especially in the problems of design of thin-walled systems (thin plates and membranes, bars and bar systems, thin-walled bars).

Timoshenko, who had used trigonometric series expansion of the deformation form, made a significant note on a possibility of restraint of only first terms of a series:

“Taking a certain shape of the bending curve, we as if introduce thereby additional constraints in our elastic system. Such an increase of the number of constraints may only be accompanied by the increase of the system rigidity, but in no way by its decrease; that is
why $P_{cr}$ values obtained by the approximate method can only exceed the real ones, but on no account be less than these values. The application of the approximate method to a whole number of problems shows that the solving accuracy, even if only one arbitrary parameter is used, is usually quite sufficient for practical application”.

An analysis of the expression for potential energy, which appears in linear problems as a quadratic functional related indissolubly to stability problem. This follows directly from the Lagrange-Dirichlet theorem. The work by P.F. Papkovych published in 1941 [Papkovych, 1941] was one of the first works where such analysis was performed in a rather general formulation. He has thoroughly considered variants of problems, when a quadratic functional of potential energy has only quadratic part or contains the terms linear in respect of generalized coordinates as well. On this basis properties of different loads from the viewpoint of their effect on equilibrium stability have been analyzed in detail. Papkovych has divided them into ordinary and special loads (active and parametrical, by terminology proposed later by A.R. Rzhanitsin [Rzhanitsin, 1955]). It has been proved that stability depends only on parametric loads and established that, owing to the effect of the parametric load, the effect made by active load on each of generalized coordinates increases $1/(1 - \varphi)$ times, where $\varphi$ is a relative value of the parametric load.

A case when the structure is subject to only one fixed kind of external effects is rather the exception than a rule. An engineer practically always deals with a set of independent loads, and these loads can act in a great variety of combinations.

P.F. Papkovych has also established general properties of critical loads for the systems subject to the action of numerous loads; he has
proved a theorem on the buckling of the area of stable equilibrium built in the load space [Papkovych, 1941]18.

For the sake of justice it should be noted that the problem of stability under multiparameter load was considered by Ludwig Föppl [Föppl, 1933]. Proceeding from the analogy between the problems of stability and theory of vibrations, and using the well-known Dunkerley formula [Dunkerley, 1894]

\[
\frac{1}{\omega^2} = \sum_{i=1}^{n} \frac{1}{\omega_i^2},
\]

He obtained for critical loads

\[
\frac{1}{N} = \sum_{i=1}^{n} \frac{1}{N_i}.
\]

Just for this reason the theorem is often called Föppl-Papkovych theorem, but, since owing to P.F. Papkovych this result has entered in the active arsenal of structural mechanics, the term Papkovych theorem is also acceptable. Later on A.R. Rzhanitsin showed that in a general case of geometrically nonlinear system the Papkovych theorem is not valid and demonstrated conditions for existence of buckling of stability area in such systems [Rzhanitsin, 1964]. Some additional results in this respect were obtained by B.M. Broude [Broude, 1964]. Properties of boundary surfaces for the cases when stability is lost in the boundary point were determined in researches by Huseyin [Huseyin, 1975], [Huseyin, 1986].

A necessity of nonlinear research of stability loss appeared in connection with the use of such structural elements as plates and

---

18 In the book [Papkovych, 1941] the author notes that his result were reported at Leningrad Mechanical Society in 1934 and were published in Iss. 1 of LKI Proceed. in 1937.
shells. The former ones, under axial compression can efficiently operate under the load defined as critical in linear analysis, and what is more, they continue working at the supercritical stage. The other ones lose suddenly stability long before achieving the “linear” critical load, the spread of experimental data being too broad. Thus the limit of linear analysis had been completely comprehended by the 40’s of the 20th century, but the nonlinear theory had not been developed.

W. Koiter solved this problem in his thesis in 1945 [Koiter, 1945]. The Koiter theory was based on consideration of small, but finite deviations from the basic (precritical) state. The most significant result was the appearance of boundary points, corresponding to the case of unstable critical state of the “ideal” simplified model. In such cases the value of the real critical load proves to be too sensitive to the effect of initial nonidealness. The Koiter theory permits obtaining asymptotic formulas, relating, the parameter of initial nonidealnesses with deviations of the value of critical load from classical one, calculated for the “ideal” structure. The energy approach has been used in the theory framework. In the vicinity of the branch point or bifurcation point the Tailor expansion of the function of the total potential energy of the system is performed with terms confinement above the quadratic order. Therewith the slope and curvature of secondary trajectory are described in the vicinity of the critical point and a conclusion is made as to the branching appearance. For example, the solution point on the secondary trajectory with a zero slope and positive curvature is identified as a stable symmetrical branch point. This allows estimating qualitatively the supercritical behaviour of the structure on the basis of studying its behavior in the vicinity of the critical point.

Nonidealness sensitivity is described as a measure of initial supercritical behavior and defined by the value of the first nonzero coefficient in power dependence of load parameter on the amplitude
of bifurcation form of stability loss. W. Koiter connects the effect of
inavoidable nonidealnesses of real structures with supercritical
behavior of the ideal structure of the corresponding form. Under
these conditions the author involves in the analysis the components
of the function of total potential energy; they are of the order above
the quadratic one.

W. Koiter’s approach permits one, first of all, to specify the idea
of system behavior near the bifurcation point that is under loads
differing little from the critical value. This approach advantage is that
the qualitative estimation of supercritical behavior of the system is
provided on the basis of its investigation in only one critical point.
The most important practical result is that the ratio between critical
loads of ideal systems and boundary loads of the systems with
nonidealnesses depends on the type of the critical point (stable or
unstable); i.e. an idea of the measure of hazard in achieving the
critical state may be formed here [Koiter, 1980].

It should be noted that W. Koiter finished his research in 1942,
when the Netherlands were occupied by Germany, and as Professor
Marcell Pignataro wrote in his obituary devoted to Koiter:

“In accordance with the occupation law, post-graduates, who
wanted to publish their thesis, had to take an oath of devotion to Nazi
government. Koiter’s dissertation dedicated to stability of elastic
equilibrium was ready, but the author, who was not going to swear to
Nazi, waited for the liberation of his country. The dissertation
appeared only in 1945”.

Koiter’s theory was applied to various structures, in particular, he
studied their nonidealness sensitivity (Fig. 3.5). Its application to the
bar, plate and membrane structure allowed revealing the causes of
difference in their behavior.

Both A. Poincare’s and W Koiter’s ideas resulted in the intention
to classify special points, but only by the 60’s of the 20th century this
approach had been developed by Thom [Thom, 1975] for the
profound classification of the systems described by the gradient
(potential) functions. It was assumed as a basis of the theory, which
was called later the theory of catastrophes [Poston, Stewart, 1980] –
the theory which gave a new vision of the stability problem

![Diagram showing load versus displacement]

Fig. 3.5. Initial disturbance sensitivity

Thompson and Hunt, implementors and propagandists of this
approach, wrote in their classical work [Thompson, Hunt, 1973]:

«... full importance of the work by Professor Koiter penetrated
slowly into our consciousness, in connection with both our
Netherland language and our characters that permanently gave an
advantage to our own studies. But saying this we have nevertheless to
note our great dependence on the work of Professor Koiter.

Though developed rather in
generalized coordinates than in
terms of continuum, our
approach is much obliged to the
above work by Professor Koiter,
and we are glad to recognize this
once more...”.

All of the above, including
the results of W. Koiter, referred
to the problems of stability of
elastic systems under the action
of conservative loads.

152
But this does not limit the circle of stability problems in mechanics of a solid deformed body. In particular studies in elasto-plastic buckling have passed a noticeable way. Euler’s approach prevailed in this sphere with the difference that a certain equivalent of elasticity modulus was used instead of the modulus itself (theory of reduced modulus).

But in 1946 there appeared F. Shanley’s solution of the problem on stability of a compressed elasto-plastic bar, that has cardinally changed the approach to analysis of stability beyond the limits of elasticity.

Shanley noticed that the conception of “reduced modulus” corresponds only to a certain single admission on load behavior and that the definition of critical force different than that used for the elastic stage should be introduced for the plastic stage.

Shanley’s work [Shanley, 1946] has shown that the existence of some forces, which are critical in the other than Euler’s sense, is possible for the elasto-plastic systems. A case, when the appearance of an allied equilibrium state is possible under the same load, is considered according to Euler, while Shanley considers a case of sustained load, i.e. the change of load. It is strange that this simple thought had not been added to armoury, the more that T. Carman used it even in 1910.

But it was Shanley’s work which became the onset for a new stage in the development of stability theory. The viewpoint of buckling beyond the limit of elasticity as the process developing with an increase of external loading was assumed as a basis of this stage.

Among the great number of works of the mentioned stage we should indicate a cycle of articles by R. Hill [Hill, 1958], where he had formulated a general condition of unity in velocities. Comparing this condition with the stability one considered as the condition of equilibrium state unity, Hill points to their nonequivalence. Hence a
conclusion is made on a possibility of bifurcation of equilibrium states under variable load (uniqueness in velocities).

Thus, as it is formulated by V.D. Klyushnikov [Klyushnikov, 1972, 1976], the phenomenon of elasto-plastic buckling is not absolutely connected with stability loss of equilibrium state, but is a result of the loss of motion stability of the body particles in the process of deformation that is a separate case of stability loss of the perturbed motion of the bodies.

V.D. Klyushnikov has also considered various processes of loading (active loading, unloading, neutral loading) and linearized problems that appear in these conditions; he has also shown in which cases the phenomenon of unloading should not be accounted under the loss of stability. Results of these researches were generalized in the monograph by V.D. Klyushnikov [Klyushnikov, 1980], where specific peculiarities of stability loss by elastolastic systems were studied on the simplest models.

Nonconservative character of the problems on elasto-plastic buckling was determined by the physical law of plastic flow. Interpretation of the loss of stability as the appearance of some new forms of equilibrium, besides the initial one, was admitted in these problems. But a more general analysis of stability of the systems being under the effect of not only conservative but also nonconservative external forces has given unexpected results even in the case of ideal elasticity. It was revealed that such systems can lose stability in such a way that the forms of stable equilibrium disappear at all. In these cases at a certain value of load, which could be called critical, as it was before, one observes the transfer not to a new form of equilibrium, but to a certain form of motion with increasing deviation from the initial position of equilibrium. The appearance of the above form of motion is the stability criterion, and this fact may be revealed only under the dynamic analysis of the problem.

This circumstance was first noticed by E.L. Nikolai under investigation of stability of a twisted bar [Nikolai, 1928, 1929]. He has found that the equilibrium configuration of a bar twisted by the
tracking or dead moment is unstable at arbitrarily small value of the twisting moment.

E. L. Nikolai reported this result at two sittings of the Leningrad Mechanical Society on May 26 and September 29, 1927. At the first sitting Nikolai’s report shocked all the present, because the idea of Euler’s critical force was well known and clear for all the audience. It is clear that everybody expected a similar phenomenon even under the effect of twisting moment: stability was to be observed at small values of the moment, and instability – when the moment exceeded some critical value. But the analysis result had shown that it was not so. And P.F. Papkovych was the first to indicate a possible cause of such strange behavior of the bar under twisting. He noted that the question is in nonconservative problems, and thus, energy can arrive to the system.

This explanation reconciled all the audience with Nikolai’s paradox. Be it as it may but the discussed information by E. L. Nikolai has initiated a new and exceptionally important section of mechanics – theory of stability of nonconservative systems. But a true boom in this theory appeared when researchers resorted to analysis of stability of the systems loaded with tracking forces [Beck, 1952], [Ziegler, 1952].

The discussions, concerning practical significance of such problems, appear sometimes in literature, many well-known scientists being involved in them. Thus a noticeable resonance was caused by a “cross” note [Koiter, 1996] on the “unreal character of tracking forces”: editors of scientific journals were recommended to reject publications devoted to analysis of the effects of tracking forces.

That could not be left out of account, the more that such authority in mechanics as Koiter had undertaken responsibility for the discussion initiation.

The most considered review of this paper was contained in two emotionally written papers of three authors [Sugiyama et al., 1999, 2002]. Admitting partial Koiter’s correctness as to the fact that there is a fatal tendency in numerous publications to purely academic style
of research without attempts to justify their applicability to practical engineering, these authors however notice a principal scientific importance of the conception of tracking forces. It could be difficult to consider various cases of dynamic instability of mechanical systems under their nonconservative loading beyond the framework of this conception. The conception of tracking forces has really given rise to lots of paradoxical results, it was difficult to understand and accept them at a glance [Boilotin, 1961]. Just in this sense the tracking forces were called “an ugly duckling of mechanics”¹⁹ in the above paper [Sugiyama et al., 2002], but in no way in slighting sence. Recollecting the fate of a hero of the tale, who turned at last into a wonderful swan, the authors cite the final words of Hans Christian Andersen:

“If (Ugly Duckling) was inexpressibly happy, but felt no pride, since a kind open heart is never proud. It remembered that time, when all mocked at him and drove him away. And now all they say that he is the best among these beautiful birds”.

3.2. Stability of compressed bars

Petrus Van Musschenbroek was the first to perform experimental system research of equilibrium stability of flexible bars under compression. Quantitative regularities found by Musschenbroek (inverse proportionality of critical force of the bar length square) were published in his work [Van Musschenbroek, 1729]. First theoretical works were made by L. Euler, who considered in 1744 [Euler, 1744] an exact equation of elastic curve (elastics)

$$\frac{C_y}{1+y'^2} = M,$$

(3.9)

And found a critical load for a column with a fixed end

¹⁹ A funny consonance of word combinations Ugly Duckling and Ugly Buckling is used with good effect in the title of the paper written in English
\[ P_{cr} = \frac{C\pi^2}{4l^2}. \]  

Equation 3.10

Euler’s name in structural mechanics is connected inseparably with the problem on stability of a rectilinear bar, though he did not search for this problem solution but came to it occasionally. He was interested in a suggestion of his friend Daniel Bernoulli that “potential force”

\[ \int \left( \frac{1}{r^2} \right) ds \]  

Equation 3.11

of a bent plate obtains the least value for a real deflection line (under constant cross-section, length and set boundary conditions). It is not difficult to see that this “potential force” differs from potential energy of a bent bar only by a multiplier \( EJ/2 \).

In 1759 L.Euler published his work [Euler, 1759], where presented the derived formulas for determining critical load proceeding from a simplified equation

\[ C \frac{d^2 y}{dx^2} = -Py, \]  

Equation 3.12

that gave again the solution (3.10). The correctness of such an approach casted doubts, and many scientists thought that the coincidence of results of the exact and linearized problems was occasional. G.-L. Lagrange allayed the apprehension; in his work [Lagrange, 1868] the researcher confirmed correctness of Euler’s formula with the help of exact analysis. He also obtained a value of critical force for a bar with hinge support at the ends, for a bar with fixed ends, as well as for a bar with variable cross-section, for which he tried to find optimal outline, though came to erroneous solution. Lagrange’s considerations were based on the thesis that to obtain load causing longitudinal bending, one should present a form of stability loss in the form of a sinusoid, and replace the length \( l \) in Euler’s formula by the distance between intersection points.

As to the form of bar stability loss (to be more precise, form of supercritical deformation), proceeding from an exact expression of
curvature, it would look as so-called Euler’s elastics and would be expressed by means of elliptical integral. This curve properties and more convenient calculation methods were given by Saalschulz [Saalschulz, 1880] and by Schneider [Schneider, 1901]. R. Mises [Mises, 1924] obtained a good approximate solution for moderately high flexibility.

In spite of the great Euler’s authority, his formulas were not recognized then by practical engineers. The fact that in the early 19th century A. Duleau showed (on the basis of his compression tests of metal bars) that Euler’s formula gives a satisfactory value of critical force, if only boundary conditions foreseen by the theory were fulfilled [Dulia, 1820]. But Dulia used comparatively thin bars, while structure elements are not so flexible in practice. For practical conditions Euler’s formula leads to exaggerated values of critical load, and further experiments were required to find the formula with a higher correspondence to reality.

A rather serious theoretical research of longitudinal bending was performed then by Lamar [Lamarle, 1845-1846]. He was the first to establish a limit to which the Euler’s formula might be used, after it one could only rely on experimental data. Lamar makes a correct conclusion that Euler’s formula may give true results until the value of critical stress exceeds the material elasticity limits, hence a flexibility limit value is found

$$\frac{l^2}{r^2} = \frac{\pi^2 E}{\sigma_t}.$$  \hspace{1cm} (3.13)

Later on, in 1887 I. Bauschinger [Bauschinger, 1887] studied experimentally stability of compressed bars. Most specimens had thin conical tips at their ends, which provided hinge support of specimen’s ends. As a result he found an increase of the specimen bending at low compressing force; it was caused by various errors. After the compressing force reached certain value, the specimen axis was considerably twisted in the plane of the least rigidity that, as a rule, resulted in the specimen fracture. The obtained value of the
compressing force prove close to critical value, that is determined by Euler’s formula, if the corresponding stress values do not reach the limit of proportionality.

In 1890 L. Tetmajer published results of experimental studies connected with stability of the compressed bars of wrought and cast iron for cases of different cross-sections [Tetmajer, 1890, 1907]. Specimens had conical tips with hinge support of the ends as it was in Bauschinger’s tests. As a result of experiments the researcher made a conclusion that, when the ratio between the bar length and minimum inertia radius is high and the bar stress is less than the material proportionality limit, the Euler’s formula is valid. An admission was made on linear dependence of the critical stress on flexibility for the case of higher stresses, the values of constants entering in this dependence were determined.

As a result of studies performed by F.S. Jasinski [Jasinski, 1895] a good agreement of experimental and theoretical results was found under stresses, which do not exceed the material proportionality limits. For cases when stresses do exceed the proportionality limit, he offered the following formula for critical stresses of the compressed bar depending on its flexibility:

\[ \sigma_{\text{cr}} = a - b\lambda \]

(3.14)

where \( \lambda \) – bar flexibility and \( a, b \) – constants which depend on material. He obtained coefficients of this dependence with the help of experimental data of I.Bauschinger, L.Tetmajer and A. Konsider by the method of the least squares.

The result of experiments, accurately performed by I. Bauschinger, L. Tetmajer and A. Konsider, was that almost 150-year doubts in Euler’s formula came to an end and there appeared interest to theoretical works in stability.

F.S. Jasinski, who had solved a number of
practically important problems of stability, mainly in bridge building practice, became one of first researchers of this “new wave”. For example, he was the first to deal with the problem on stability of compression braces of a bridge truss with a cross lattice with allowance for supporting effect of tension braces [Jasinski, 1894]. Jasinski considered a number of cases of stability of the compressed bars under the effect of distributed loads. In particular, he considered a bar with hinged ends in elastic medium, which response was perpendicular to the bar axis and proportional to its deflection, while intensity of the compressing distributed axial load was proportional to distance from the middle section.

This problem was called “Jasinski’s problem”, he had reduced the calculation of compressed top cords of open bridges [Jasinski, 1894]. This study was initiated by a series of accidents of such bridges in Western Europe and in Russia.

Jasinski was the first to generalize formulas of critical stress that allows reducing any case of longitudinal bending to the main case of hinged bar by substituting a reduced (design) bar length instead of a real bar length in the formula. Introduction of the notions of the coefficient of length and design length of the bar was of great importance for all further development of the methods for buckling analysis of compressed bars, since it allowed applying formulas and test data, obtained for the basic case, to any case of longitudinal bending. Following Jasinski, other scientists also used this idea in their researches, and it was included in science as one of the main characteristics of the compressed bar under its buckling analysis.

S.P. Timoshenko, Jasinski’s pupil and follower, already in his work of 1907 [Timoshenko, 1907] presents an exact and approximate solution for the bar, lying on continuous elastic Winkler type basis; in his next publication Timoshenko [Timoshenko, 1907] considers stability of a bar with elastically fixed ends and stability of a cantilever of variable cross-section. He estimated approximately the role of cross-section weakening on a small part of the bar length and considers a question on the shear effect on critical force. In the latter
case the Nissbaum formula [Nissbaum, 1907] was obtained by the energy method.

\[ \frac{1}{P_{cr}} = \frac{1}{P_{cr,e}} + \frac{1}{P_{cr,s}}, \]  

(3.15)

where \( P_{cr,e} = \pi^2 EI / l^2 \) is the critical force under the bend buckling, \( P_{cr,s} = GA/k \) – critical force under the shear form of stability loss, \( k \) – cross-section form coefficient.

Some other problems of stability of a rectilinear compressed bar were also solved, but the problem of design of short compressed bars losing stability at inelastic stage of operation, which arose then before theoreticians was the most important one. Only the above said empirical dependences were applicable to them.

In 1885 F. Engesser [Engesser, 1885] offered using in Euler’s formula a tangential elasticity modulus \( E^* \), determined by the slope angle of a tangent to a curve deformation-stress: \( E^* = d\sigma/d\varepsilon \), when operating with material beyond the proportional limit.

F. Engesser thought that in the moment of stability loss under the bar bending the compressive strains of the cross-section part on the concave side of the bar increase, and on the convex side – decrease. Therewith the tangent module \( E^* \) serves as a proportionality coefficient between strain and stress increments.

Later on F. Engesser answered criticism of F.S. Jasinski, who had noticed that an ordinary elasticity modulus was to be used under the unloading, taking place on the cross-section outside. In 1889 F. Engesser published a revised formula for the critical force:

\[ P^{**} = \frac{\pi^2 E^{**} l}{l^2}, \]  

(3.16)
where $E^{**}$ is a reduced modulus of elasticity (the name appeared later), which depends both on the tangent modulus $E^*$, and on the unloading modulus $E$, as well as on the section shape.

The same formula was obtained in 1910 by T. Karman, whose thesis [Karman, 1910] contains the expression for the reduced modulus in a case of rectangular cross-section:

$$E^{**} = \frac{4E^*E}{(\sqrt{E} + \sqrt{E^*})^2}. \quad (3.17)$$

In contrast to F. Engesser, who departed from Euler’s formula that fixed the stability loss condition, T. Karman first considered a problem on the longitudinal bar bending as a problem of its buckling and stability in the process of monotonously lasting loading. For physically and geometrically nonlinear problems the stability loss occurred in the Poincare limit point. T. Karman confirmed his results by thoroughly made experiments over short bars with preset initial imperfectness in a form of eccentricity of the compressing force.

The Engesser-Karman approach, based on the use of the reduced modulus, remained the only one under consideration of longitudinal bending of inelastic bars during the whole first half of the 20th century.

However, in most experiments critical loading, determined by the tangent modulus $E^*$, was closer to reality than reduced-module loading $P^{**}$. In 1946 Shanley described the formed position as a certain paradox [Shanley, 1946]. He made his own experiments and found that at initial stages of buckling the increment of strains on the concave side much exceeds the decrease of strains on the convex side of the bar. Thus the buckling beginning is connected with a zero volume of the unloading zone, and calculation should be made with the use of tangent modulus. Besides, it appeared that the ceasing of the load increase conserves the bending, i.e. stability loss takes place only in conditions of loading increase [Shanley, 1947].

Karman notes in his commentaries to Shanley’s paper:
“My primary research, as well as that of Engesser, is a generalization of the theory of elastic buckling. Why does not it embrace all possible equilibrium positions in inelastic case? It is evidently not because in inelastic region the ratio between stress and strain is nonlinear, but because the deformation process is of irreversible nature. Depending on the history of the process of loading and unloading there are a lot of values of residual strain that correspond to one and the same stress. So, for irreversible processes the determination of stability limit is to be revised. Shanley has intuitively recognized this necessity that, I suppose, is the great merit of his work.”

Yu. M. Rabotnov [Rabotnov, 1952] also confirmed that Shanley’s solution was true, if the increase of compressing load (a conception of lasting loading) proceeds simultaneously with the bending. Shanley has actually indicated the lower limit for compressing force; new forms of fixed member equilibrium can appear, beginning from this limit. From this viewpoint the tangent-module force may be considered as critical.

The use of tangent-module critical loading is attractive from the practical point of view, and its relatively low difference from reduced-module critical loading (Fig. 3.6) removes a lot of doubts.

Shanley’s idea on the influence of the loading history on the value of critical loading has led A.A. Ilyushin [Ilyushin, 1960] and V.G. Zubchaninov [Zubchaninov, 1965] to creation of the stability theory of bars that allows for interaction of the latter with the structure under buckling. A.A. Ilyushin showed that the buckling of a compressed bar, operating as a component of structure of low rigidity, may be accompanied both by a decrease of loading on a bar (unloading structure), and by increase (loading structure). V.G. Zubchaninov has made analysis of the process of postbifurcation buckling of bars in unloading and loading systems of arbitrary rigidity and has shown that there is a whole spectrum of bifurcation loadings with stable and instable postbifurcation buckling.
3.3. Eccentrically compressed bars

A design model in a form of absolutely straight centrally compressed bar is the abstraction. It had long been known and it was quite natural to aspire for estimating such abstraction. Thomas Young was the first here as well as in numerous problems in mechanics.

In the second volume of Young’s course of lectures published in 1807 [Young, 1807] he considers a problem of bending of a compressed column which has a primary curvature in a form of a sinusoid $f_0 \sin (\pi x/l)$. A deflection of a cantilever free end caused by the compressing force $P$, proves equal to

$$f = \frac{f_0}{1 - \left(\frac{4Pl^2}{EI\pi^2}\right)}.$$  \hspace{1cm} (3.16)

Hence it is concluded that at $P = \pi^2 EI/4l^2$ the deflection acquires the infinitely big value, whatever be the value $f_0$. That was the first studying of a pillar with the availability of initial bending.

Navier solved this problem in 1819. Analogous results were also obtained for other cases of the bar compression work with bending.
For example, in 1913 S.P. Timoshenko performs an approximate solution for calculation of bending of a compressed-bent elastic bar in a form:

\[ f = \frac{f_0}{1 - S/S_e}, \]

where \( f_0 \) – a deflection from only lateral load, \( S \) – compressive force, \( S_e \) – Euler’s critical force. Note that it is a separate case of the problem of interaction of active and parametric loadings considered in a general form by P.F. Papkovich.

It is seen from this formula that, when a compressive force approaches Euler’s critical force, the deflections begin tending to infinity that was interpreted as the loss of stability. And it was concluded that the lateral loading does not change the critical force of the bending buckling of a compressed (centrically or eccentrically) rectilinear bar.

Ratios of such type were then often used for constructing various approximate solutions of the problems on stability and stable strength. In particular, R. Southwell, based on this dependence, has developed his method of experimental determination of critical loading that allows for possible nonidealness of the tested specimen [Southwell, 1932].

It will be instructive to note that the strength control is connected with a formula:

\[ \sigma = \sigma_N + \frac{Nf_0}{W} \cdot \frac{1}{1 - \sigma_e/\sigma_N} + \frac{M}{W} = \]

\[ = \sigma_N \left( 1 + \sigma_M \frac{c l^2}{h \rho} \cdot \frac{1}{1 - \sigma_N/\sigma_e} \right) + \sigma_M = \sigma_N (1 + \nu) + \sigma_M, \]

where \( \sigma_N = N/A \) — compressive stress, \( \sigma_M = M/W \) — bending stress, \( \rho \) — kernel distance, \( c \) — coefficient in the expression for determining the bending deflection \( f_0 = cMl^2/(Wh) \).
And if the value \(1/(1+v)\) is identified with coefficient of lateral bending \(\varphi\), we obtain the well-known Jasinski’s formula:

\[
\sigma = \sigma_N / \varphi + \sigma_M = \frac{N}{\varphi A} + \frac{M}{W}.
\]  \hspace{1cm} (3.19)

The values of coefficient \(\varphi\) and the values \(1/(1+v)\) differ in general by 5-7%, but the main defect of Jasinski’s formula is that real stresses \(\sigma_M\) are added to fictitious stresses \(\sigma_N / \varphi\). In spite of this the Jasinski formula is used sometimes even now.

First researchers of the question connected with eccentric compression of inelastic bars faced difficulties related to a necessity to consider various cases of distribution of elastic and plastic region in the cross-section; as a result, the solution became dependent on this cross-section shape. The problem on determining a critical force of eccentrically loaded bars was first studied by T. Karman as the stability problem. Under these conditions the buckling of bars was foreseen in a plane passing through one of the main symmetry axes of the cross-section. Based on his theory E. Chwalla investigated carefully the stability of eccentrically compressed bars in a number of papers published in the period from 1928 to 1937, for example [Chwalla, 1934]; he generalized the results obtained for different cross-section shapes in a form of tables and diagrams.

Chwalla’s solution depends on the diagram of material compression \(\sigma - \varepsilon\) taken by him, and this diagram had a short yield area, the self-reinforcement stage coming quickly then. This circumstance, as well as the fact that E. Chwalla’s solution was tedious and was distinguished by an immerse graph-analytic form of calculation, prevented from using this method in practical calculations. But it should be considered that this work gave impetus to creating simpler methods of calculation. Beginning from the 30’s just this trend attracted the greatest number of researchers in numerous countries.

Jezek [Jezek, 1937] made a valuable contribution to this problem solution. He solved the analytical stability problem beyond elasticity.
limits of a hinged eccentrically compressed bar, using a model of the ideal plastic material (Prandtl diagram). The use of one diagram eliminated a necessity to consider different diagrams for each material and allowed developing general methods for structures design. This diagram clearly differentiated the elastic and plastic stages of the structure work. Owing to this he justified the notion of a plastic hinge that was of great importance for understanding the often observed cases and concerned the essential plastic deformations of the structure with preserving its carrying capacity.

One more admission was that the bent line of a bar was taken as a sinusoid half-wave. This admission proved completely justified, and approximate methods of stability calculation of compressed-bent bars based on this admission are rather accurate from the practical point of view.

A.R. Rzhanitsin, taking a hypothesis of plane sections, had expounded a similar method of determining critical states of eccentrically compressed bar on the curve length-deflection under a preset compressing force [Rzhanitsin, 1955]. Based on the same approaches, other researchers also proposed the simplified methods of stability calculation for compressed-bent bars. And at last, taking the admission about of the bar axis twisting as the sinusoid half-wave, on the material, subject to the Prandtl diagram, as well as considering equilibrium equation for the central cross-section related to the compressing force and moment, one can obtain an expression for the extremal value of the mean cross-section stress for main cross-sections depending on flexibility and eccentricity. Approximate formulas and tables of coefficients were obtained for determining critical force beyond the elasticity limit of eccentrically compressed bars, which are widely used in the present Building codes.
3.4. Stability of a plane bending form, bending-torsion form of buckling

The problem of the side buckling of beams of narrow rectilinear cross-section was first considered by Prandtl [Prandtl, 1899] and Michell [Michell, 1899]. They published in 1899, independently of one another, the theory of side buckling of the beams under the effect of lateral loading, and both came to the same result – to differential equation of the problem, which is a second order homogeneous equation with variable coefficients. A nontrivial solution of this equation gave a critical value of the lateral load, which bends a band.

Further development of this problem belongs to S.P. Timoshenko, who obtained in 1905-1906 the fundamental differential equation of torsion of symmetric H-beams and studied on this basis a side buckling of the loaded high H-beams [Timoshenko, 1905-1906]. Timoshenko’s research was later presented for defense for the scientific degree of adjunct in applied mechanics at Kyiv Polytechnic Institute (his opponents were V.L. Kirpichov, A.A. Radtsig and M.B. Delone). And the main thing is that this research has turned the Prandtl academical problem into the problem, being of great significance for bridge-building practice.

The work by Timoshenko [Timoshenko, 1905-06] gave impetus to numerous studies of other authors. Thus, a thorough research of the effect of elevation or lowering of the load point by the value \( P_{cr} \) was performed by A. Korobov [Korobov, 1911]. The H-beams with different flanges under simultaneous effect of axial compression and equal end moments in the wall plane were first considered by Friedrich Bleich [Bleich, 1933]. This theory was also applied to the problem on calculation of H-beams, which extended flanges were anchored opposite to a lateral shift, i.e. to the problem of buckling with a fixed rotation axis.

In 1941 Winter [Winter, 1941] presented approximate formulas for the problem of side buckling of nonsymmetrical H-beam, which he had obtained basing on the use of Rayleigh strain-energy method.
Investigation of the problem on side buckling was performed by Goodier [Goodier, 1942], who had obtained solution of the general problem of stability of thin-walled bars of open cross-section under simultaneous effect of compression, bending and torsion. Goodier does not introduce any constraints connected with the cross-section symmetry, but considers only the effect of longitudinal forces and end moments. The work touches the questions concerning beams, being under the effect of lateral load.

In the context of the above mentioned problem the question of torsion of thin-walled members of open cross-section has assumed practical importance. The simplest case of stability loss of a torsion form of angular cross-section was considered by S.P.Timoshenko [Timoshenko, 1905-06]. G. Wagner [Wagner, 1929] made a general investigation of stability loss of a torsion form of thin-walled members, similar to those used in aircraft structures. A more strict justification of this theory belongs to R. Kappus [Kappus, 1937]. Over the time that had passed since publication of these works, a lot of engineers worked over studying the lateral buckling of beams and torsion form of stability loss of compressed thin-walled members. Results of these researches are widely used not only in aircraft industry, but also in bridge construction.

We should note here the works by Goodier [Goodier, 1942], who studied stability of not only a separate compressed bar in different conditions but also a bar rigidly joined to elastic plates. Using the theory of great deformations, he has given a strict support of the actual correctness of the precondition, assumed as a basis for the theory of stability loss in torsional form, developed by G. Wagner. G. Nylander [Nylander, 1943] enriched the theory of lateral buckling of H-beams and performed the important experimental research on this problem. E Chwalla [Chwalla, 1941] studied lateral buckling of the beams of asymmetrical profile and presented a general form of equations, the equation for H-beam being their particular case.

S.P. Timoshenko has expounded a general theory of bending, torsion and stability of thin-walled members of the open cross-
section [Timoshenko, 1945]. V.Z. Vlasov [Vlasov, 1949] has developed in his book another, more general approach to the theory of thin-walled bars, characterized by bending-torsional form of stability loss. As for such bars, each element of cross-section displacement (two shifts and torsion) has its corresponding critical force. The lateral force being applied to the bending centre, these forces become independent and the least of them becomes a decisive one. The analysis made by V.Z. Vlasov has shown that the critical force assumes the highest value, if it is applied to the bending centre. The compression force being applied to other cross-section points, including the centre of gravity, the critical force decreases. V.Z. Vlasov revealed the role of cross-section deplanation, which is especially characteristic of the cross-sections of open cross-section. He has shown that the critical force increases under deplanation limitation.

3.5. Stability of curvilinear bars

Moris Levy [Levy, 1884], was the first to consider such type problem. He has found critical loading distributed uniformly all over the length of the circle of incompressible ring.

Following Levy the overwhelming majority of solved problems on equilibrium stability of curvilinear bars rests on the condition of incompressibility of the bar axis (Boussinesq condition). The acceptance of Boussinesq condition permits essential simplification of mathematical posing of the problem. Different variants of load behavior were analyzed.

M. Levy has found a critical value of load \( q_c = \frac{3EI}{R^3} \), and a more strict analysis of this solution showed that it is exact, if the load is considered hydrostatic, i.e. follows in its direction the normal to strain surface of a curvilinear bar. For the ring compressed by the
dead radial load, which remains parallel to initial direction even after the loss of stability, the solution looks as \( q_{cr} = 4EI/R^3 \). For the polar load (such a case appears under cooling the rim of the wheel with many spokes) E.L.Nikolai [Nikolai, 1918] obtained \( q_{cr} = 4.5EI/R^3 \).

Timoshenko [Timoshenko, 1923] studied stability of a supported unfixed-end circular bar under uniform radial pressure and obtained exact solution of this problem. A case when loading (even after stability loss and remaining parallel to initial circle plane) was directed to the ring centre.

A lot of works were dedicated to the problem on stability of arches. A question of the acceptance of a hypothesis on the arch axis incompressibility added a noticeable complexity. Suppose that this simplifying condition will be used only for too high-rise arches. In case of shallow arches the assumption on the arch axis incompressibility may lead to considerable calculation errors which increase with a decrease of the ratio of the arch rise and its chord. The point is not only that one can neglect the axial deformation, when determining the lateral force in a shallow arch. There is another aspect of the problems of equilibrium stability of shallow arches. In these problems one should call into question even admissibility of the problem linearization, especially intensive linearization, when the initial state of the system is considered stressed but not strained.

But the abandonment of linearization means that from the very beginning the problem was to be mathematically considered in nonlinear posing, and it is clear that this complicated both a mathematical posing of the problem and its solution. However the approximate solution of these problems may be obtained. For example, the behavior of a shallow sinusoidal arch under the effect of vertical loading was considered by S.P. Timoschenko [Timoschenko, 1935].
This S.P.Timoshenko’s research was continued by A.N. Dinnik, who had published a big cycle of works on arches stability, made in 1930-40’s [Dinnik, 1946]. In particular, he noticed one essential circumstance that had escaped S.P. Timoshenko’s attention. The question is that in a certain range of parameters of a shallow arch the loss of its equilibrium stability happens not after reaching the limit point on the curve of equilibrium states, but somewhat earlier and by the bifurcation criterion. Though Dinnik made a correct and important conclusion, determination of the value of a critical bifurcation loading for a shallow arch was incorrect.

3.6. Stability of plates

The history of development of stability theory for plates under the effect of edge compressive forces started in 1891, when Bryan [Bryan, 1891] had published his research on a rectangular plate freely supported at all edges under the action of compressive load located in the plate centre and uniformly distributed at opposite margins.

The buckling problem has attracted the particular attention fifteen years later. In addition to Bryan’s results (he obtained solution for a hinge-supported plate uniformly compressed in one direction or compressed in two perpendicular directions) investigations were started on stability of rectangular plates with different limit conditions and with different loading diagrams. S.P. Timoshenko has considered a problem of buckling of rectangular plates under different conditions of support of the edges parallel to acting compressive forces, including a variant of a plate, when one of its unloaded sides is free. [Timoshenko, 1907].

Independent of Timoshenko Reissner had published solution of the problem on a rectangular plate with two fixed ends as well as with one fixed and one free end under the effect of compressive edge forces [Reissner, 1909]. Such a variant of loading allows studying local stability of a thin-walled bar with angular cross-section.
Stability of a hinge-supported at longitudinal edges of the infinite band compressed by oppositely directed concentrated forces $P$ was studied in 1907 by Sommerfeld [Sommerfeld, 1907], who obtained an exact solution of this problem. Different authors often returned to this problem, since it had become a control one.

In his course of ship-building mechanics I.G. Bubnov first solved a problem on stability of a rectangular plate, where a couple of opposite sides is loaded by compressive and tensile forces changing linearity along these sides [Bubnov, 1912]. This solution allows considering stability of the wall of cross-section of the bending beams. There was also a problem on stability of a rectangular hinge-supported plate under the action of shearing stresses uniformly distributed along the plate contour.

In 1915 S.P. Timoshenko returned to the problem of plate stability. Now he studied the effect of the plate reinforcement by ribs [Timoshenko, 1915].

In 1924 Friedrich Bleich made an attempt to extend the stability theory of plane plates to inelastic problems, considering a plate as anisotropic body and introducing a variable elasticity modulus in the main differential equation, which served as the solution basis that corresponded to elastic buckling [Bleich, 1924].

Ros and Eichinger [Ros, Eichinger, 1932], Bijlaard [Bijlaard, 1940] and Ilyushin [Ilyushin, 1944] made an attempt to formulate a rational stability theory of plates beyond the elasticity on the basis of the theory of plasticity, but their results did not give an exact coincidence with results of experiments. And only Stowell, using Ilyushin’s general ratios, succeeded in developing a rational theory of inelastic buckling; the results of this theory are in good agreement with laboratory observations [Stowell, 1952].

Compared with the theory of elasticity of compressed bars, the problem of stability of plates is complicated by the circumstance that
the value of critical loading may differ from the value of limit loading carried by the plate. In contrast to a bar the increase of the critical state of a plate is not always equivalent to the exhaustion of its carrying capacity, since in supercritical state at displacements compared with the plate thickness there occurs redistribution of chain forces which is favorable for the plate work. It should be noted that these effects are not realized under any limiting conditions. Two main qualitatively different cases of supercritical behavior are possible for a plate as well as for a compressed bar. If the restraints of the plate contour do not prevent its general purely bending deformation, i.e. the deformation without elongations and displacements of the middle plane is possible after the loss of stability (Fig. 7, a), the plate behavior after the loss of stability will be the same as the behavior of a bar with butt ends unfixed in respect of longitudinal displacements.

Flexible plane plates differ from other numerous elements of the structural complex, since their loss of stability cannot be often identified with the limit state. A possibility to use their work at the supercritical stage sharply distinguishes the lamellar elements, and estimation of admissible extent of entering in the supercritical area is an important scientific-and-technical problem, a lot of researches being devoted to its solving.

![Diagram](image)

Fig. 3.7. Two cases of supercritical behavior

Practical importance of research of supercritical work of a plate bonded at the edges with rigid ribs and compressed in one direction, has led to a necessity of creation of a simplified engineering method of calculation, based on the idea of reduction factor.
The idea of the reduction factor, as a ratio between the critical stress and the stress in rigid constraints, was introduced by I.G. Bubnov in respect of calculation of the ship hull and was developed by P.F. Papkovych, P.A. Sokolov, T. Karman, K. Marger and a number of other researchers.

A simple approximate formula for the reduction factor, which does not pretend to allowance for all the circumstances of supercritical plate deformation was obtained in the work by T. Karman [Karman et al., 1932], \( \eta = \sqrt{\sigma_{cr}/\sigma_p} = \sqrt{1/n} \), where the relation of the stress in the limit fiber to critical stress is marked through \( n = \sigma_p/\sigma_{cr} \).

Under the plate shear work the stability loss is connected with appearance of goffering in the direction of the main compressive stress. Under these conditions the plate fibers, parallel to the formed folds, can sustain essential tensile forces transferred to the contour ribs. Thus a system of oblique tensile forces is formed; they are balanced by the reaction of compression in transversal ribs, which appear as the truss pillar. Such a model, proposed by G. Wagner [Wagner, 1929], was called a diagonal-extended field.

3.7. Stability of shells

First fundamental results in the problem of shell stability were obtained in the early 20\textsuperscript{th} century ([Lorenz, 1908, 1911], [Timoshenko, 1914], [Southwell, 1913-1915]) in a linear posing on the basis of L. Euler’s statical criterion. Idealized design diagram was used in these works. A shell was considered geometrically perfect and ideally elastic, with momentless initial state. Such a posing was then widely used and became a classical one. The value of critical load obtained in these works was not corroborated by first experiments. Critical loads observed in experiments were, as a rule, much lower than the theoretical ones. All the further development of the theory of shell stability was directed to revealing the causes of this divergence.
Later on Donnell [Donnell, 1934] turned his attention to the importance of allowing for nonlinear terms in geometric ratios, though the conceptual questions of this problem, which concerned stability loss of bars and spherical dome due to the flip, had been discussed earlier in the works by I.G. Bubnov, S.P. Timoshenko and Biceno. The notions of the upper and lower critical loads were introduced in a wonderful paper by S.P. Timoshenko [Timoshenko, 1925]. They were especially broadly used after the work by Karman and Tsien [Karman, Tsien, 1939].

Karman and Tsien [Karman, Tsien, 1941] established that the load falls in supercritical zone with the increase of strain. That was unexpected and was in contrast with all results obtained, when solving similar problems for bars and plates, where loading continuously increased with the increase of strain. A sharp decrease of load after change of the output unperturbed equilibrium form evidences for availability of nonadjacent bending forms of equilibrium at low levels of loading and extraordinary shell sensitivity to various disturbances. The value of the upper critical load proved dependent on the type and measure of perturbations (mainly due to the imperfect form of the middle surface). The lower critical load that determines stress level in the shell (other equilibrium forms, except for output one, cannot exist below this level) came into use for estimating stability of shells.

The deflection function was approximated by a trigonometric series, the small number of the series terms being retained then. The state has somewhat changed with involvement of electron-digital machine (computer). There appeared a possibility to specify solution, increasing the number of degrees of the shell freedom. As a result it was found that the lower critical load decreases with an increase of the number of terms, confined in the expansion of searched functions. What is more, the negative values of the lower critical load were obtained in some works. This and some experimental works have changed the viewpoint as to a lower critical load as a feature of shell stability.
The theory of stability at that stage was mainly developed in breadth at the expense of studying different classes of shells and different kinds of load. One of the basic trends of research was a detailed study of behavior peculiarities of shells under load. A typical result of this trend of works was the description of spherical domes [Bushnell, 1981] depending on the shallowness parameter

$$\gamma = 2 \left( 3 \left( 1 - v^2 \right) \frac{H}{t} \right)^{1/2},$$

where $H$ – a rise, $t$ – shell thickness.

![Diagram of load-deflection curves for different shallowness parameters.]

**Fig. 3.8. Effect of shallowness parameter**

At $\gamma = 0$ (plane round plate) the curve “load-deflection” is characterized by continuously growing rigidity connected with the appearance of tension in the plate centre. At $\gamma \leq 3.5$ the curve “load-
deflection” has not a horizontal tangent anywhere and stability loss is not realized. At $\gamma \leq 6$ an impact proves possible in the limit point, but no bifurcation loss of stability appears. At $\gamma > 6$ there appears a bifurcation transfer to asymmetric form of stability loss that appears before the symmetric flip in the limit point, and at $\gamma > 7$ the bifurcation comes not only before the symmetrical flip, but also before the classical critical load of a closed sphere.

A trend connected with the study of the effect of initial shell imperfectness on the value of the upper critical load roused great interest. The effect of initial imperfectness of the shell shape was first mentioned in the work by Flügge [Flügge, 1932], where he explained physical essence of the phenomenon and posing of the linear problem. Additional terms, which characterize a change of a curvature due to the initial deflection, were introduced in linear equations of stability.

In 1945 Koiter [Koiter, 1945] studied in detail the behavior of various elastic systems near the bifurcation point. The dependence for critical stresses was obtained for the simplest axisymmetric form of initial deflection with wavelength equal to that at the perfect shell stability loss; according to this dependence even small axisymmetric incorrectness leads to considerable decrease of the critical force. Donnell and Wan [Donnell, Wan, 1950] had finally formulated the method of allowance for shell imperfectness which was widely used with time. In accordance with this method all the initial shortcomings (geometrical, physical, etc.) are accounted by introduction of a certain equivalent initial deflection similar to the deflection because of stability loss.

The method of initial imperfectnesses qualitatively collects phenomena observed in the experiment, and allows (by choice of the corresponding amplitude of initial deflection) obtaining experimental critical load. But there is certain indeterminedness in choosing the amplitude of initial deflection, since the value and form of initial imperfectness are of occasional nature. A statistical analysis of
results of experiments in studying initial imperfectnesses is required for practical use of this approach.

Such posing of the problem was probably first precisely formulated in the work by V.V. Bolotin [Bolotin, 1958], where statistical stability theory of shells was based on the admission that a common density of probabilities of occasional parameters of the problem (including those describing initial imperfectnesses) is known. Almost simultaneously with this publication there appeared a work by I.I.Vorovich (Vorovich, 1959), which, in contrast to Bolotin’s approach, was based on the theory of Markov’s processes and on the admission that generalized coordinates of the problem possess Markovian properties. Such an approach was further developed in the works by V.M. Goncharenko, [Goncharenko, 1962] and M.F. Dimentberg [Dimentberg, 1962].

As to analysis of the effect of initial imperfectnesses, the first attempt to realize Bolotin’s methodology was made by his postgraduate student B.P. Makarov, who, using the experimental data on distribution of critical loads, found the law of distribution of the parameters characterizing initial imperfectnesses [Makarov, 1962]. Bolotin’s methodology was also used and specified in the works of Thompson [Thompson, 1967] and Roorda [Roorda, 1971].

Important matter has been plaid by the paper [Elishakoff, 1979], where for the first time the method for accounting of stochastic initial imperfections for the problem of column buckling on nonlinear elastic foundation has been proposed.

This method has been applied successfully for structural behavior analysis of shell with sensitivity to small stochastic shape imperfections. It also has been developed intensively by researchers
from Delft University of Technology [Arbocz, 1981], [Elishakoff, Arbocz, 1982], [Koiter at all., 1994] and from other scientific centers.

Numerical studies of shell stability were performed during the last decades mainly with the help of the method of finite elements. These works are characterized by attempts of ever more “realistic” modeling of behavior of the structure under load, including a detailed nonlinear analysis. A principal difference of the linear and nonlinear behavior of shells is revealed under these conditions.

This is illustrated by Fig. 3.9, taken from [Schmidt, Swadro, 1997]. For a double cone (Fig.9, a) the nonlinear load of buckling is only 40% of the linear one, though the pictures of strain look rather similar. For the system cylinder-cone-cylinder (Fig. 9, b) the corresponding relation is 65%, forms of stability loss are also different in this case.

Fig. 3.9. Comparison of results of linear (LA) and nonlinear (GNA) calculations

3.8. Stability of multielement elastic systems

A bar (including a thin-walled one) or plate is most frequently a part of a more complex structural system. That is why such problems as behavior of a compressed bar, being a truss part, or stability of the whole system of a rigid frame are the most important ones for a
building engineer. Rigid joints of bar system elements are the cause of the fact that the bending of one element in the state of buckling causes deformation of other elements of the structure. Each element is bound with others by elastic restraint, and the stage of restraint of any element depends on bending rigidity and axial load of all the other. Hence it is clear that the study of stability is required for obtaining real condition of buckling of the whole system or for satisfactory analysis of real behavior of a compressed element or certain group of such elements, entering into the bar system.

Analogous remarks could be introduced as to multielement systems which include not only bars but also plates and shells. But in all cases here and further we will tell about the systems composed of separate elements studied beforehand, independent of whether they are bars of a certain bar framework or a system of finite elements representing nonbar part of the structure.

**Bar systems**

The simplest (more complex than a single bar) system is a straight bar with intermediate rigid or elastic supports. The problem on such system buckling had been first studied by Engesser, who derived an approximate formula for any preset value of rigidity of elastic supports [Engesser, 1884]. Engesser considered the compressive force as constant in length, and the effect of elastic support in discrete points he changed by the action of continuous elastic base. Then Jasinski considered in the same way the case of the loading bar variable in length [Jasinski, 1894].

The problem on calculation of the bar system stability had been considered by Zimmermann [Zimmermann, 1909, 1910], who considered stability of a compressed multispans bar with rigid or elastic supports. He obtained this problem solution in a form of a determinant. Similar solution was obtained by Müller-Breslau [Müller-Breslau, 1908], who studied stability of the upper boom of an open bridge. This problem had been already solved by F.S. Jasinski, but he did it by approximate presentation of the design
model as a beam on elastic base, i.e. by spreading the elastic support of transversal frames. But the methods used in these works were oriented to a straight multispans bar and had no required universality.

The important result was obtained in 1912 by I.G. Bubnov [Bubnov, 1913], who thoroughly analyzed the problem on stability of a continuous bar on elastic supports and established that, beginning from a certain value of the increase of support rigidity does not affect the value of critical load. Therewith the elastic supports become equivalent to absolutely rigid ones. Then the problem was also formulated and solved, which concerned stability of the system of parallel bars compressed by equal forces and supported in a span by cross-located elastic bars. The latter problem, called the Bubnov problem, gave impetus to other numerous studies. When solving a problem of continuous bar, I.G. Bubnov used the reactions of elastic supports as the basic unknowns, thus using first the method of forces in the study of stability.

In 1919 Friedrich Bleich published a systematic research of stability of plane bar systems with rigid nodes, conducted in respect of general solution of the problem for all types of framed structures.

Bleich thought that, if the equilibrium state under study is considered side by side with a new state close to it, the external load and axial forces in the elements, being in equilibrium immediately before the transfer from the stable to unstable position, do not already form a balanced group of forces, since the external forces change their position (not a direction) at the expense of deformation of the system as a whole. That is why additional internal forces are formed in the system, and the ratios between additional forces and moments and displacements of nodes form the basic equations, the stability conditions proceeding from them. Physical content of these equations is compatibility of deformations.

Additional longitudinal forces and end moments in bars, displacements of nodes and distortions of bars are unknowns in these equations, while a coefficient at these unknowns are the functions of structure dimensions and axial forces \( P \), determined by loads
affecting the structures. The method is based on the idea of renewal of the system of linear relations between the moments in nodes and rotation angles of the bar. These two groups of parameters are sufficient to describe in a single way any form of unstable state of the frame.

The first group of stability equations expresses the condition of continuity in the node of joining of two or more elements. When considering a node, \( n \) bars entering in it, \( n - 1 \) couple of bars is considered by turns (Fig. 3.10), the condition \( \varphi_i + \alpha_i = \varphi_j + \alpha_j \) is fulfilled for each of them.

Using the dependence among the rotation angles, moments at the bar ends and longitudinal force, these equations may be given the following form, for example, for a pair 1-2

\[
M_1^i \frac{l_s}{I_2} + M_1^i \frac{l_s}{I_1} + M_2^i \frac{l_c}{I_1} + M_2^i \frac{l_c}{I_2} + M_2^i \frac{l_s}{I_2} - E(\alpha_i - \alpha_2) = 0, \quad (3.20)
\]

where \( s_i = \frac{1}{v_i^2} \left( \frac{v_i}{\sin v_i} - 1 \right) \), \( c_i = \frac{1}{v_i^2} \left( 1 - v_i \cot v_i \right) \), \( v_i = \sqrt{\frac{P_i}{EI_i}} \).

![Fig. 3.10. On deriving the stability equations](image-url)
Presenting a frame as a set of \( m \) closed contours \( F \). Bleich records the condition of preservation of closeness of each contour

\[
\sum_{i \in j} \frac{P_i}{EA_i} \cos \gamma_i + \sum_{i \in j} \alpha_i l_i \sin \gamma_i = 0 \quad \sum_{i \in j} \frac{P_i}{EA_i} \sin \gamma_i + \sum_{i \in j} \alpha_i l_i \cos \gamma_i = 0
\]

where \( \gamma_i \) — slope of the \( i \)th bar towards the horizon.

At last, the equilibrium equation which binds the moment at the beginning \( M_b \), moment at the end \( M_e \) and transversal force \( Q \) may be recorded for each bar

\[
M_{i,b} = M_{i,e} + (Q_j \cos \gamma_i + H_j \sin \gamma) l_i .
\]

Here \( Q_j, H_j \) \( (j = 1, \ldots, m) \) — transversal and longitudinal forces in the contour cross-section.

The system of equation (3.20), (3.21), (3.22) relative to unknowns \( M_i, \alpha_i, Q_j, H_j \) is homogeneous, since the external forces do not change. By force of homogeneity and linearity of stability equations they are satisfied by zero values of node displacements, and the nonzero form of deviation is possible, only if the above system determinant \( \Delta \) is equal to zero. Determinant \( \Delta \) is the function of load, which may be characterized by the intensity parameter \( \lambda \) that is the unknown value in conditions of stability \( \Delta = 0 \).

Owing to introduction of the notion of reduced modulus it proved possible to use the theory in the elastic and plastic areas of buckling. In the work of 1926 [Bleich F., Bleich H., 1926] F. Bleich and H. Bleich have extended this approach to the problem of stability of spatial bar systems.

The question of general stability of hinged bar systems, which buckling is realized on straight bars, was considered in general form by Mises and Ratzersdorfer [Von Mises, Ratzersdorfer, 1923]. In their other work Mises and Ratzersdorfer [Von Mises, Ratzersdorfer, 1925] expounded in detail the problem on bar systems with rigid nodes, also extending the research to the systems in which the
change in elements length, determined by axial forces, affects the condition of stability. This research is mathematically identical to the work [Von Mises, Ratzersdorfer, 1923] and differs from the latter only by the choice of the type of the basic moment ratios, which it is based on.

The solution by Mises and Ratzersdorfer for trusses was essentially improved by Ya.L. Nudelman in 1942 [Nudelman, 1942]. He noticed that, if the node load is decomposed, its components being oriented along the bars, then in the perturbed equilibrium state the two forces, acting at the ends of any bar, form a couple, which moment is proportional to the bar rotation angle. The equations of equilibrium of these moments are homogeneous, and their determinant being put to zero, one can find the critical load.

The works [Bleich, 1919], [Bleich F., Bleich H., 1926] were based on the fact that stability equation is made concerning the given problem, proceeding from its peculiarities. But in 1936 V. Prager developed a method for studying stability of bar systems, repelling from the analytical condition of stability of a compressed bar with the elastically turning and displacing nodes [Prager, 1936]. This work was a direct precursor of the idea of applying the displacement method to the problem of stability of the bar systems – the idea, proposed in 1937 simultaneously and independently by A.A. Bilous [Bilous, 1937], N.V. Kornoukhov [Kornoukhov, 1937], and S.D. Leites [Leites, 1937], and independently used in 1941 by Chwalla and Jokisch, [Chwalla, Jokisch, 1941]. The idea might fly in the air, and its sense was that the coefficients of canonical equations of the displacement method were calculated using formulas which represented the dependence of reactions to the value of longitudinal force in the bar under consideration.

Usage of the approach based on the method of forces and proposed in the works by V.G. Chudnovsky [Chudnovsky, 1952] and S.D. Leites [Leites, 1949], proves less efficient.

For the displacement method the rigidity matrix of a compressed bar
\[
\mathbf{R}(v) = \\
\begin{bmatrix}
\frac{EI}{l^3} \gamma(v) & \frac{EI}{l^2} \delta(v) & -\frac{EI}{l^3} \gamma(v) & \frac{EI}{l} \delta(v) \\
\frac{EI}{l^2} \delta(v) & \frac{EI}{l} \alpha(v) & -\frac{EI}{l^2} \delta(v) & \frac{EI}{l} \beta(v) \\
-\frac{EI}{l^3} \gamma(v) & -\frac{EI}{l^2} \delta(v) & \frac{EI}{l^3} \gamma(v) & -\frac{EI}{l^2} \delta(v) \\
\frac{EI}{l^2} \delta(v) & \frac{EI}{l} \beta(v) & -\frac{EI}{l^2} \delta(v) & \frac{EI}{l} \alpha(v)
\end{bmatrix}
\]  

depends on the dimensionless parameter of compression

\[
v^2 = \frac{NI^2}{EI\left(1 - \frac{N}{GF}\right)}
\]  

(3.24)

through M.V. Kornoukhov’s function

\[
\alpha = \frac{cv^2 \tan\left(\frac{v}{2}\right)}{2(2\tan\left(\frac{v}{2}\right) - cv)} + \frac{v}{2\tan\left(\frac{v}{2}\right) - cv}, \quad \beta = \frac{cv^2 \tan\left(\frac{v}{2}\right)}{2(2\tan\left(\frac{v}{2}\right) - cv)} - \frac{v}{2\tan\left(\frac{v}{2}\right) - cv},
\]

(3.25)

\[
\gamma = \frac{c^2 v^3}{2\tan\left(\frac{v}{2}\right) - cv}, \quad \delta = \frac{cv^2 \tan\left(\frac{v}{2}\right)}{2\tan\left(\frac{v}{2}\right) - cv}.
\]

In the framework of displacement method the bar system itself is considered as a set of its nodes, mated by elastic bars. Under these conditions all the degrees of freedom prove to be concentrated just in the nodes, and a system of canonical equations of displacement method looks as follows

\[
\mathbf{R}_c(v) \mathbf{Z} = 0.
\]

(3.26)

The condition of existence of a nonzero solution of this system of equations, which is formulated by the known algebraic demand

\[
\det\left[\mathbf{R}_c(v)\right] = 0
\]

(3.27)

Just is the analytical criterion of the approach of the system critical state.
3.9. Search of critical load

All the above approaches were based on the fact that critical load value is determined from the condition that a determinant of the system of solvable equations is equal to zero, and the form of stability loss corresponds to the least root of the determinantal equation. Thus, from the viewpoint of mathematics, the equilibrium stability problem has been reduced to the eigenvalue problem.

Numerous investigations were dedicated to a search for a convenient method of solving this mathematical problem, even when authors use “purely mechanical” considerations. The well-developed theory of linear homogeneous differential equations, playing a decisive part in the small vibration theory, proved very useful. There is a complete analogy between the small vibration theory and the theory of elastic stability that is explained by availability of mathematical foundation of the both problems.

Further research may be imagined as two parallel flows. The first trend dealt with development of the methods of eigenvalues (critical forces) finding, and the second trend – with so-called qualitative methods, which do not require to determine critical forces, but allow estimating the equilibrium quality (stable or unstable) for each loading level.

When considering various methods of solving stability problems, one can see three considerably different approaches:

- Direct analytical consideration of the question by deriving and solving linear homogeneous equations relating the axial forces, displacements and rotations of rigid nodes as well as the moments acting in them. The stability criterion is expressed by a certain determinant equality to zero.

- The energy method based on energy criterion of stability also leads to the system of linear equations. The stability criterion takes again the form of zero equality of the corresponding determinant.
• Qualitative method based on estimation of the arrangement of trial value of the load intensity parameter in the spectrum of critical forces of the system under study.

The two last approaches were especially popular. S.P. Timoshenko transformed the energy method into a powerful method for studying various too complicated problems of buckling. Just at the same time W. Ritz [Ritz, 1909] published a classical paper, where he had developed on the broad mathematical basis a general method for direct solution of the problems on minima in mathematical physics. He used this method for studying the equilibrium and vibrations of rectilinear plates with four clamped edges. The use of the both methods (Ritz’s and Timoshenko’s ones), when solving stability problems, leads to the same mathematical transformations and to results similar as to mathematical form.

No essential changes have been introduced in this problem solution up to 1935, when Trefftz developed (in addition to the method of Ritz) the method for determining the lower limit of the critical force [Trefftz, 1930]. The Ritz method in its primary form gives higher approximate values of the critical force than the exact value, and thus, determines the upper limit of the critical force. So, the Trefftz method allows bounding the problem solving by the upper and lower limits that is too important, when estimating the solution accuracy.

As to determining the load intensity at which the determinant of the system of solving equations turns into zero, it appeared that this problem is reduced to the eigenvalue problem. The eigenvalues were determined by the methods (which follow from the mechanical sense of the problem) by A.M. Krylov [Krylov, 1931], P.F. Papkovych [Papkovych, 1933], S.A. Bernshtein [Bernshtein, 1940] et al. Iteration methods by Von Mises and Pollaczek-Geiringer, [Von Mises, Pollaczek-Geiringer, 1929], Lundquist [Lundquist, 1939] were rather often efficient.
3.9.1. Qualitative methods

A. Poincare may be considered a founder of qualitative approach to stability problem. He offered to distinguish the equilibrium quality (stable, unstable) under the effect of the set load by signs of coefficients in a canonical presentation (3.2) for potential energy of the system. But this approach was precise only for the systems with the finite number of the degrees of freedom. Substantiation of the approach extension to bar systems, having the infinite number of the degrees of freedom, required additional forces.

Using the method of displacements (as well as other methods of structural mechanics) created the illusion of transfer to the system with the finite number of the degrees of freedom, since researchers considered the finite number of unknown displacements or forces.

The continual part of the problem proved to be hidden in the system of specifying functions of the type (3.25); these functions are used to calculate coefficients in the rigidity matrix and compliance matrix. It was necessary to study the expressions of potential energy of the system, properties of coefficients which help construct these expressions, properties of eigenvalues and eigenvectors of the rigidity matrix and compliance matrix.

One of the first such works was the above work by P.F. Papkovych [Papkovych, 1941]; a lot of questions were settled by A.F. Smirnov, who was probably the first to present qualitative analysis in a direct form [Smirnov, 1947].

This method had practically obtained a civic right, when it was indicated that the reduction of quadratic function of potential energy to the sum of squares (3.2) was not a necessary procedure, and diagonal elements of the matrix of canonical equations reduced to upper triangular form may be used as stability coefficients instead of $\alpha_i$.  

Anatoliy Filippovich Smirnov  
(1909—1986)

The expression for bar system energy was investigated in this work in a general form, and it was shown that this expression may be presented as a sum of two summands, the first of them belonging to the system with the finite number of freedom degrees, and the second – to the system obtained from the preset one under imposing the ban on those displacements, through which the first system energy was expressed. It appeared that, if the second system (basic system) is stable, only system stability coefficients with the finite number of freedom degrees may be used to estimate equilibrium quality.

If it is impossible to guarantee stability of the basic system (for example, when setting a trial value of the load parameter $\nu$, exceeding a critical value), then, as it turned out, the critical value $\nu_{cr}$ of load parameter may be missed, when using this determinant criterion (3.27). The matter is that the determinant criterion, as to its content, traces only such forms of equilibrium stability loss, when the displacement vector $\mathbf{Z}$ differs from the zero vector. But it can happen that the forms of stability loss of the basic system (emphasize, not an original system, but the basic one) may include such forms which do not produce forces in all the introduced constraints – these forces prove to be equal to zero. That means that such forms are homogeneously proper both to the set system and the basic system of the method of displacements. Then, the determinant criterion will not be capable of catching these, so-called latent forms [Lyakhovych, 2004]. The corresponding basic system of displacement methods is called a wrong one [Smirnov, 1947]. The error connected with using
such basic system can be avoided, if Euler’s critical forces of certain bars of the basic system, which form additional stability series, are considered jointly with the stability series. It should be also said that the notion of the wrong basic systems as well as latent and explicit forms of stability losses is not a prerogative of the method of displacements. The situation analogous, as to its content, to the wrong basic system of the method of displacements is quite possible both in the method of forces and in the mixed method.

S.P. Timoshenko was probably the first, who had noticed the existence of latent forms when using the method of forces. Considering the problem on stability of a multi-span bar he indicated [Timoshanks, 1905-06]:

“The system of equations obtained in such a way can give non-zero solutions for the support moments, only if this system determinant becomes zero. It is the determinant equality to zero that will give us the equation required to find the critical load. ... Zero equality of all the support moments, corresponds either to direct form of bar equilibrium or to the case, when all the spans under twisting bend independent of each other, i.e., when a compressive force in each span equals the critical force of this span”.

Another useful result, following from qualitative analysis, is connected with the fact that a single element or a small group of elements is often in fault of stability loss of the whole structure. In this connection the notions of the state of limited and forced loss of stability of separate parts of the structure with the general loss of stability were introduced even in the 40’s in the works by M.V. Kornoukhov [Kornoukhov, 1949], and A.F Smirnov [Smirnov, 1947]. But only recently, A.V. Perelmutter and V.I. Slivker [Perelmutter, Slivker, 2001] and O.V. Aleksandrov [Aleksandrov, 2001] practically simultaneously and independently indicated in their works the criterion of determining the bifurcation form (limited of forced) of a bar or of any part of the structure.
3.9.2. Numerical methods in stability problems

The appearance of effective numerical methods, the method of finite elements in particular, caused a break in numerical studies of stability problems. The finite-element formulations of the theories of structural members on the one hand, and the effective algorithms of numerical investigation of stability of carrying structures, on the other hand, have been developed.

Instability points are most often determined within the framework of MCE in solution of a linearized eigenvalue problem at complete neglect of deformations and stresses in the precritical state. To allow for the available stress the same linearized problem is solved, as a rule, but at the stress-strained state of the structure determined as a result of linear calculation. Such methods are included in numerous finite-element programs with the function of stability analysis.

It is necessary to allow for all the previous history of nonlinear load for the nonlinearly operating structures. That became possible after rather precise methods of nonlinear deformation analysis had appeared, e.g. such as step-iteration methods of continuation of the curve of equilibrium states along the arc.

The engineering approach to defining critical points of nonlinear problems consists in completing the step-iteration methods of solution by so-called “accompanying actions” directed to investigation of the structure behavior stability [Brendel, 1979]. For example, the determinant value of the rigidity matrix or the number of its negative diagonal elements is traced [Krftzig, Qian, 1991], [Wagner, 1991]. The eigenvalue problem may be solved in the vicinity of the critical point. However, all these algorithms do not allow obtaining the exact value of the critical point. The so-called “bisection method”, based on linear interpolation of tangent matrices of neighboring configurations, is involved, as a rule, to specify solution, when passing through the critical point. One of modifications of the above method is described in the work [Belytscko et al., 2000].
The method of finite elements in combination with the step-iteration algorithms is also used to determine supercritical behavior of the structure. To do this one should not only calculate the critical points, but also define their type, which corresponds to Koiter’s classification [Koiter, 1967]: limit point, asymmetric branch point, symmetrically stable and symmetrically unstable branch point. In the framework of finite-element approach to classification of singular points the authors usually rest on accompanying operations [Wagner, Wriggers, 1988].

Reliable methods of solution continuation for cases of passing through the limit point have been developed for studying the structure behavior after the onset of critical state. The solution continuation in bifurcation points presents essential calculation difficulties. Here, to study the transfer to secondary trajectory one uses the disturbance of the equilibrium state, when in the branch point vicinity of nonlinear equation the latter is substituted by the certain number of the terms of Taylor’s series expansion [Gulyaev et al., 1982], [Eckstein, 1983].
References

Bolotin, V.V. (1961), Nekonservativnyie zadachi teorii uprugoi ustoichivosti [Nonconservative problems of the theory of elastic stability], Fizmatgiz, Moscow, Russia.
Bubnov, I.G. (1913), Stroitelnaya mekhanika korablya, Chast 1 [Ship-building mechanics, Part 1], Izdanie morskogo ministerstva, St. Petersbourg, Russia.
Vlasov, V.Z. (1949), Stroitelnaya mekhanika tonkostennych prostranstvennykh system [Structural mechanics of thin-walled spatial systems], Stroiizdat, Moscow, Russia.
Dimentberg, M.F. “Nonlinear vibrations of elastic panels at random loads”, Izvestiya AN SSSR. OTN, Mekhanika i mashinostroenie, no. 5, pp. 102-110.
Dinnik, A.N. (1946), Ustoichivost arok [Stability of arches], OGIZ, Moscow-Leningrad, Russia.
Klyushnikov, V.D. (1980), Ustoichivost uprugoplasticheskikh system [Stability of elastoplastic systems], Nauka, Moscow, Russia.
Kornoukhov, N.V. (1949), Prochnost i ustoichivost sterzhnevykh system [Strength and stability of bar systems], Stroyizdat, Moscow, Russia.
Kornoukhov, N.V., Varvak, P.M., Rakovitsan, P.M., Strelbitskaya, A.I. and Chudnovsky, V.G. (1938), Issledovanie ustoichivosti prostranstvennogo karkasa po tipu vysotnoi chasti Dvorta Sovetov [Study of stability of spatial framework by the type of high part of the Palace of Soviets], Izdatelstvo AN Ukr.SSR, Kiev, USSR.
Lyapunov, A.M. (1935), Obshchaya zadacha ob ustoichivosti dvizheniya [General problem on motion stability], OGIZ, Leningrad-Moscow, Russia.
Lyakhovich, L.S. (2004), Razdeleniye kriticheskikh sil i sobstvennykh chastot uprugikh sistem [Separation of critical forces and eigenfrequencies of elastic systems], Izdatelstvo GASU, Tomsk, Russia.
Moiseyev, N.D., (1949), Ocherki razvitiya teorii ustoichivosti [Essays on development of stability theory], Gostekhteorizdat, Moscow-Leningrad, USSR.

195


Nudelman, Ya. L. (1949), Metody opredeleniya sobstvennykh chastot i kriticheskikh sil dlya sterzhnevykh system [Methods for determining eigenfrequencies and critical forces for the bar systems], GIITL, Moscow, USSR.


Poston, T. and Stewart, I. (1980), Teoriya katastrof i yeyo primeneniy [The theory of catastrophes and its applications], Mir, Moscow, USSR.


Rzhantinsin, A.R. (1955), Ustoichivost ravnovesiya uprugikh system [Equilibrium stability of elastic systems], Gostekhizdat, Moscow, USSR.


Smirnov, A.F. (1947), Staticheskaya i dinamicheskaya ustoichivost sooruzhenyi [Statitical and dynamical stability of structures], Transzheldorizdat, Moscow, USSR.


Timoshenko, S.P. (1915), Ob ustoichivosti plastinok podkreplennykh抵御 пуста [On stability of plated reinforced by elastic ribs], Izdaniye Institututa putei soobshcheniya, St.Petersburg, Russia.

Hill, R. (1958), Obscheaya teoiiya edidstvennosti i ustoichivosti dlya uprugo-plasticheskikh tel [General theory of uniqueness and stability for elasto-plastic bodies], Mekhanika (Sbornik perevodov), no.3 (49), pp. 53-65.

Chudnovsky, V.G. (1952), Metody raschota kolebanii ustoichivosti sterzhnevykh system [Methods of calculation of vibrations and stability of bar systems], Izdatelstvo AN Ukr.SSR, Kiev, USSR.


Bleich, F. (1933), Die Knickfestigkeit elastischer Stabverbindungen, Der Eisenbau, Vol. 10, p.27.


197


Chwalla, E. (1934), Theorie der aubermittig gedruckten Stabes aus Baustahl, Stahlbau, no. 21-23.

Chwalla, E. and Jokisch, F. (1941), Über das ebene Knickproblem des Stockwerkrahmens, Der Stahlbau, Heft 14, S. 33


Engesser, F. (1884), Die Sicherung offener Brücken gegen Ausknicken, Zentralblatt der Bauverwaltung, S. 415


Euler, L. (1740), Scientia Navalis seu Tractatus de Construendis ac Dirigendis Navibus, Typis Academiae Scientiarum, St. Petersburg, Russia.


Föppl, A. (1920), Drang und Zwang: Eine höhere Festigkeitslehre für Ingenieure München, R. Oldenbourg Verlag, Berlin, Germany.

Föppl, L. (1933), Über das Ausknicken von Gittermasten, insbesondere von hohen Funktürmen, Zeitschrift für Angewandte Mathematik und Mechanik, Band 13, Heft 1, pp. 1-10


Saalschulz, L. (1880), Der belastete Stab, Leipzig, Deutchland.


Tetmajer, L. (1907), Die Gesetze der Knickungs und zusammengesetzten Festigkeit der technisch wichtigsten Baustoffe, Leipzig.


Thom, R. (1975), Structural stability and morphogenesis, Reading: Benjamin.


Van Musschenbroek, P. (1729), Introductio ad cohaerentiam corporum firmorum, Lugduni.


Essay 4

APPEARANCE AND FORMATION OF THE CALCULUS OF VARIATIONS
Calculus of variations – qualitatively new one, different than differential calculus...

L. Euler
4.1. First variational problems

The last quarter of the 17th century and early 18th century are characterized by quick development of infinitesimal calculus. After H. Leibnitz had initiated it in European science, and I. Newton laid the foundations of infinitesimal calculus in England, the infinitesimal calculus is obliged for its development to the works by the members of Bernoulli’s family of mathematicians [Wipper, 1875, Fleckenstein, 1949, Spiess, 1948], mainly to Jacob and Johann Bernoulli.

Four representatives of the Bernoulli family: Johann (1667-1748), his sons Daniel (1700-1782), Nicolas (junior, 1695-1726) and Daniel’s nephew – Jacob junior (1759-1789), a propo, married to L. Euler’s granddaughter – were honored members and professors of mathematics and mechanics of the St Petersburg Academy of Sciences.

Jacob I Bernoulli (1655-1705) considered a problem on the shape of a bending curve of elastic bar. While Gallilei and Mariotte studied the beam strength, Bernoulli solved the problem of the deflexion calculus20.

Johann I Bernoulli (1667-1748), Jacob’s younger brother was considered an outstanding mathematician of his day. His teaching activities resulted in the first book in infinitesimal calculus written by marquise De l’Hopital [De l'Hospital, 1696]21 in 1696. Johann Bernoulli (in his letter to Varignon22) presented first strict formulation of the principle of virtual displacements. He was L. Euler’s teacher. He has posed and solved the problem on brachistochrone, laid, jointly with J. I Bernoulli, the principles of variational calculus, was a founder of mathematical physics.

---

20 The first draft approach to this problem was published in Leibniz’s journal «Acta eruditorum Lipsiae» in 1694 p.; the final treatment was give by the author in Histoire de l'Académie des sciences de Paris in 1705.

21 There is Russian translation [L'Hopital, 1935].

22 See: [Varignon, 1725, P. 174].
Daniel Bernoulli (1700-1782) is known as the author of the book *Hydrodynamics*, but he also favored the development of researches in the theory of elastic curves.

J.J.Bernoulli – a Swiss mathematician, is the most known representative of the family, as a younger brother and pupil of Jacob Bernoulli, leading mathematician in Europe of the 18th century, teacher of H.F.A. l’Hopital and L. Euler. He has posed and solved the problem of brachistochrone, laid, jointly with J.J.Bernoulli the principles of the variational calculus, was a founder of mathematical physics. He argued the priority in posing the variational problem with Jacob. His scientific correspondence was about 2500 letters.

Even in Gallilei’s works we find a problem, which preceded the problem on brachistochrone. He was the first to raise a question of the curve of descent in his *Two New Sciences* in 1638. “Theorem XXII, proposition XXXVI: if from the lowest point of a vertical circle, a chord is drawn subtending an arc not greater than a quadrant, and if from the two ends of this chord two other chords be drawn to any point on the arc, the time of descent along the two latter chords will be shorter than along the first, and shorter also by the same amount, than along the lower of these two latter chords”, that is Gallilei’s Theorem means only that motion along the ar of a circle is quicker than along the corresponding chord or any inscribed broken line.

In 1696, in a June book of Leipzig journal *Acta eruditorum* (P. 269) J.J.Bernoulli had published a note *New Problem – Mathematician Are Invited to Solve It*. That was the problem on brachistochrone. The solution of (in H.Leibniz opinion) “such a wonderful unprecedented problem” was presented by J.Bernoulli, H.Leibniz, I.Newton, J.Bernoulli and H.l’Hopital. Ja.Bernoulli gave half a month for its solution. But only H.Leibniz sent his solution, as time had passed. J.J.Bernoulli extended the term till the Easter 1697. In this period the problem was solved by I.Newton, Ja. Bernoulli and H. l’Hopital.

Note that the solution of the problem on brachistochrone by J.J.Bernoulli (1697) concerns simultaneously optics and mechanics, i.e. running of a ray and a heavy material point. Indicating that
P. Fermat obtained the law of refracted light from the principle of the shortest time (at \( v = \text{const} \)). Fermat’s principle of the shortest time is transformed into the principle of the shortest path), J. Bernoulli considers a problem on the ray curvature in inhomogeneous transparent media. J. Bernoulli notes: “Thus, I have simultaneously solved two problems – one optical, other – mechanical; that is I have done more than I required from others... Before completing I cannot help but express once more my surprise as to the marked unexpected identity between Ch. Huygens’ tautochrone and our brachistochrone. What is more, I think it necessary to mark that this identity issues from the basic propositions of G. Galilei; hence we could conclude that this proposition is in agreement with nature. The nature always acts in the simplest way, and in this case it presents two different services using one and the same line”. Here J. Bernoulli means that, as Ch. Huygens has shown, a cycloëde being in the vertical plane in such a way that the line of its basis is horizontal and lies above the the acting circle, has the property – whatever point on this curve this body begins to descent from, it comes to the lowest position during one and the same time (tautochrone).

Soon after the work of J. Bernoulli on brachistochrone there appeared (and were solved) a lot of problems of the same type.

In 1697 J. Bernoulli had sent one more extremal problem to Journal des savants: to draw the shortest line between two preset points at arbitrary surface. First studies were conducted by Leibniz
and Ja. Bernoulli, but the most important result was obtained by J. Bernoulli himself. He established that at any point of the shortest line the half-tangent plane is perpendicular to tangent plane of the surface that, as is known, is the main property of geodetical lines. J. Bernoulli informed H. Leibniz in his letter of August 26, 1698. It is unknown how he has come to this conclusion. Investigation of geodetical lines has extended a class of variational problems and simultaneously favored development of analytical geometry.

First edition of the book by J. Bernoulli *Opera omnia, tam antea sparsim edita*. 1742

J. Bernoulli set his pupil Leonard Euler (1707-1783) a problem to find a general approach to their solution. L. Euler’s work *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrii latissimo sensu accepti* (Method for Finding Curves Having Properties of Maximum or Minimum, or Solution of Isoperimetric Problem Considered in the Broadest Sense) appeared in 1744 [Euler, 1744]; in this book theoretical principles were laid of a new section of mathematical analysis. In particular, approximating curves by broken lines, L. Euler derived differential equation of the second order, which is to be satisfied by extremals. In due course Joseph-Louis Lagrange (1736-1813) called it Euler’s equation. In 1759 there appears the first work of J.-L. Lagrange and
new methods of research. J.-L. Lagrange “varies” a curve suspected for extremum, isolates from increments of functionals main linear parts which he calls variations and relies on the fact that in the point of extremum the variation is to be zero. The Lagrange method soon becomes commonly accepted. Note that after the works by J.-L. Lagrange on L. Euler’s proposition the whole section of mechanics, to which the Lagrange method was applied, was called the calculus of variations.

It is considered that when (about 1700) I. Newton, Ja. I Bernoulli, J. I Bernoulli, G.F.A. ’Hopital and H.V. Leibniz solved the problem on brachistochrone, which had been formulated in 1696, J. I Bernoulli initiated the development of variational calculus (theory of functional extremums). It is of interest that the symbol of variation was introduced by Lagrange, though the term variation was introduced by L. Euler much later, and the term functional – by G. Hadamard only in 1903.

In 1742 in the letter of October 22 Daniel Bernoulli suggested L. Euler, a pupil of J. Bernoulli, the idea of using the variational calculus to obtain the equations of elastic curves: “Since nobody has mastered such a perfect use of isoperimetric method (variational calculus) as you, you will easily solve the problem in which it is necessary that \( \int_0^L \frac{ds}{\rho} \) acquired the least value”\(^{23}\). That was the first posing of variational problem in the history of elasticity theory. Evidently, the presented functional with the accuracy to \( \frac{1}{2} EI \) – potential energy of elastic deformation. L. Euler published the proof of this proposition in 1744.

\(^{23}\) See: Fuss P.H. «Correspondence mathematique et physique» (Correspondence in mathematics and physics) [Fuss, 1843]. See: Fuss P.H. «Correspondence mathematique et physique» (Correspondence in mathematics and physics) [Fuss, 1843].
Daniel Bernoulli was the first to obtain differential equation for transversal vibrations of a prismatic beam, studied partial cases of vibrations, performed the great number of experiments.

Euler in his book *Methodus inversendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti* (a propo, that was the fist book on variational calculus) approaches the problem solution from the viewpoint of calculus of variations. He notes: “Since the world structure is perfect and erected by the sage (the Creator), nothing in the world occurs where a sense of any maximum or minimum were noticed, so it is out of doubt that all the phenomena may be successfully determined both for the final causes using the method of maxima and minima and for the causes themselves... That is there are two ways of understanding the natural phenomena: one – through determining causes, which is called a direct method, another – through the final causes, and mathematics uses successfully both of them. But first of all additional efforts should be directed to solution using the both methods; only then, not only one solution is confirmed by another, but we get the highest delight from agreement of the both” 24.

To illustrate these approaches L. Euler considered the problem on catenary.

A curve presenting the equilibrium state “by direct method” may be obtained for a chain (Fig. 4.1). In so doing we consider forces affecting its infinitesimal element $mn$ and equilibrium equations are derived for these forces. A differential equation of catenary is derived from the obtained equations. The same aim may be reached by the

---

24 English translation of the book Appendix, dedicated to the study of elastic bending lines was made by W.A. Oldfather, C.A. Ellis and D.M. Brown. See: *[Isis, 1933, P. 1]*. See also German translation in the series «Ostwald's Klassikei» («Oswald’s Classics»), № 175). Russian translation [Euler, 1934, pp. 447-572].
“method of final causes” (by Euler’s terminology), attacking the problem from considerations of potential energy: gravity forces. From all possible deflexion curves the searched one should be such, for which its potential energy is of the least significance, or, and that is the same, the equilibrium curve is one, for which the centre of the chain gravity occupies the lowest position. Thus, the problem is reduces to a search for the extremum of functional \( \int_{\sigma}^{s} w \cdot y \, ds \), where \( s \) – the set length of the curve, \( w \) – weight of the chain length unit. Using the method of variational calculus we come to differential equation.

Passing to the case of elastic bar, L. Euler notes that the “direct method” was used by Ja. I Bernoulli. When using the “method of final causes” L. Euler uses the data of D.I Bernoulli from the letter of October 22. He writes: “Daniel Bernoulli, venerable and most witty in this high kingdom of nature, has informed me that he can present all force contained in a bent elastic plate by one formula, which he calls “potential force”, and that this expression for the elastic curve should be the least”, and then continues (according to Bernoulli): “… if only a plate is uniformly thick, broad and elastic, and is rectilinear in natural state”, for this case “the expression \( \int_{\sigma}^{s} (1/R^2) \, ds \) value will be the least one”. Using his variational calculus Euler obtains a differential equation of Jacob Bernoulli for an elastic line, for a cantilever loaded by force \( P \) at the end, in a form:

\[
\frac{Cy^4}{(1+y'^2)^{3/2}} = Px. \tag{4.1}
\]

L. Euler does not restrict himself by consideration of only small deflections, he integrates this equation by series expansion.
L. Euler was a mathematician, mechanicist, physicist and astronomer, one of the founders of variational calculus (works of 1727-1741). In 1744 there appeared his work *Method of Finding Curved Lines* – the first book on variational calculus. He has written memoirs in almost all fields of mathematics and mechanics. The list of his works consists of 850 titles, including multivolume monographs. Complete works by L. Euler in 72 volumes were published in 1909 in Switzerland. Besides, his correspondence, which numbers 3000 letters, was published only partially. They said that L. Euler did not like theatre and, if got there, yielding his wife’s persuasions, he did complex mental calculations to avoid a bore; he found the amount of calculations to last the spectacle. Two students had once fulfilled individually complex astronomic calculations; the obtained results differed in the 50th sign and they asked L. Euler for help. The latter had mentally made the same calculations and gave a correct result.

In 1739 there appeared L. Euler’s work *Tentamen novae theoriae musicae* in mathematical music theory. There was a joke as to this work that it contained too much music for mathematicians and too much mathematics for musicians.

In the 1730’s twenty-year old L. Euler was engaged in studying isoperimetric problems. Having several works in solving such problems L. Euler published in 1744 a treatise, where he had included all his former results, which concerned isoperimetrical problems, and presented a so-called direct method for finding the curves, providing extremum of undetermined integral of a certain form.

On August 12, 1755 Euler received unexpectedly a letter from 19-year J.-L. Lagrange from Turin, who had filled (just in the letter) main omissions in L. Euler’s considerations. G.-L. Lagrange introduced a more distinct difference of two kinds of “differentials”, fixed the symbol $d$ for signing the basic part of the function at the expense of the change of argument and proposed a new symbol $\delta$ for changes determined by passing from one curve $y(x)$ to the comparative one. Instead of guessing correlations between the values
\( \delta y \) and \( \delta dy \), as Euler did it, admitting the mixing of two kinds of “differentials”, Lagrange deduced these ratios, resting on a possibility of permutation of the symbols \( d \) and \( \delta \) (without substantiation of this property). Thus, firstly, Lagrange had introduced a new symbol – a variation symbol (with no name) and proved a possibility of permutation of the operations of differentiation and variation:

\[
\delta dF(y) = d\delta F(y), \\
\delta d^2F(y) = d^2\delta F(y) \quad \text{and so on}
\]

(4.2)

and, secondly, Lagrange offered, in analogy with ordinary differential calculus, to equate the first variation of functional \( J \) to zero, considering this as the necessary condition of its extremum

\[
\delta\int Zdx = 0, \quad \text{aбо} \quad \int \delta Z = 0, 
\]

(4.3)

\[
\delta Z = N\delta y + P\delta dy + Q\delta d^2y + R\delta d^3y + \ldots,
\]

(4.4)

where

\[
N = Z'_y, \quad P = Z'_dy, \quad Q = Z'_d^2y, \quad R = Z'_d^3y,
\]

\[
\int N\delta y + \int P\delta dy + \int Q\delta d^2y + \int R\delta d^3y + \ldots = 0.
\]

(4.5)

Integration by parts gives:

\[
\delta\int Z = \int (N - dP + d^2Q - d^3R + \ldots)\delta y + (P - dQ + d^2R - \ldots)d\delta y + (Q - dR + \ldots)d^2\delta y + \ldots = 0.
\]

Since all the curves started and ended at the same points, Lagrange equated all the terms outside the integral to zero and obtained the well-known Euler’s equation

\[
N - dP + d^2Q - d^3R + \ldots = 0. 
\]

(4.6)

L Euler was surprised by the depth of thought of the Turin novice. When developing principles of the variational calculus he left one problem, since he had not proved the ratio

\[
PdP + pdP = 0.
\]

He expressed a displeasure in this respect: “…We need one more method, free of geometrical means, for finding maximum or minimum, we have to write \( pdP \) instead of \( PdP \)” [Euler, 1958, P. 116].

215
In this connection J. D’Alembert writes: “To make the motives, which evoked enthusiasm of L.Euler, more noticeable we should refer to the origins of numerous researches of J.-L. Lagrange, which he pointed two days before his death. The first attempts to determine maxima or minima of all undetermined integrals were made in relation to the curve of the quickest descent (brachistochrone) and isoperimeters of Bernoulli. These researches were done again by L. Euler, who presented a general method in the original work, where the depth of analysis mastering was observed everywhere. But great as this method might be, it did not possess such a simplicity that is desired in the work of pure analysis. The author understood it himself, he thought it necessary to find a proof independent of Geometry and Analytics [D’Alembert, 1867].

That was made by J.-L. Lagrange. In his letter to L. Euler of October 20, 1755 J.-L. Lagrange considered the problem on brachistochrone with a mobile end, when the end point of the extremum was not fixed, but could belong to a certain line, which character was set by the boundary conditions (in present terms these are the so-called natural boundary conditions).

The first memoir by J.-L. Lagrange on variational calculus Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indefinies (Experience of a New Method for Determining Maxima and Minima of Undetermined Integrals) appeared in Turin Notes in 1760-1761.

L. Euler has made much to emphasize essential deserts of J.-L. Lagrange in creation of a new field of mathematics – variational analysis. Presenting the young scientist with a possibility to publish his achievements, L. Euler retarded publication of his own results in the same field for several years. When he had published his revised method and expounded it in analytical form (like it was made by J.-L. Lagrange) he indicated again J.-L. Lagrange’s deserts in the Introduction: “After I had made long and fruitless work on settling this question, I noticed in surprise a simple and happy solving of this problem in the Turin Notes. This wonderful discovery enraptured me,
since it differs considerably from my methods and is much more simple”.

Euler’s actions were too timely: J.-L. Lagrange’s memoirs caused doubts and misunderstanding, as Fontain and Bordat wrote. A. Crelle wrote about that a bit later in notes to German translation of the course of lectures by J.-L. Lagrange Lectures on Function Calculus.

Main vaguenesses arose because of J.-L. Lagrange’s statement that the function \( y(x) \), which provides extremum of the integral \( I \), changes by the rules of ordinary differentiation, though this change should be distinguished by a special index \( \delta \) instead of ordinary \( d \).

L. Euler answered all these questions and doubts, explained the essence of a new variational calculus, gave it name and developed a lot of additions of the method.

“Calculus of variations, - L. Euler wrote, - is qualitatively new, different from differential calculus…” And then: “And curves that infinitely little differ from the unknown one, are most suitably considered as those obtained under the increase or decrease of ordinates of certain points of the required curve by infinitesimal values, that is under ordinates variation. Usually it is suffice to perform such variation for the only ordinate, but there is nothing to prevent from ascribing such variations to several or all ordinates, since we always have to come to the same solution. But in so doing not only the method force is revealed to a higher extent, but more complete decisions for such kind questions are obtained…”.

The order of operating on variations is too similar to differentiation rules; they have much in common, but the most significant analogy is that in both cases infinitesimal increments are added to variables. But the essential difference between two calculi should be remembered: “when we deal with a curve compared with that close to it, we pass, using differentials, from one point of the curve to other points of the same curve; while passing from this curve to another one, similar to the former, and if this transition is infinitesimal, it is realized with the help of variations…”.
Thus, L. Euler has explained the obscure statements of “differential calculus” by J.-L. Lagrange, operating on symbol $\delta$. Variations of values $\delta y$ mark nothing else but infinitesimal increments of the values $y$ at the expense of transition from one curve to another, infinitesimally similar.

In the work *Mécanique analytique* (Analytical Mechanics, 1788) [Lagrange, 1788], J.-L. Lagrange summed up everything made in the field of mechanics during the 18th century.

Lagrange assumed a principle of possible displacements as a basis of statics, and a combination of the principle of possible displacements with J. D’Alambert principle – as a basis of dynamics. In the works *Theory of Analytical Functions* (1797) and *Lectures on Function Calculus* (1801) he made an attempt to substantiate the analysis, reducing it to algebra. He has offered analytical expounding of the variational calculus. Proceeding from L. Euler’s results, he developed the basic ideas of variational calculus. In the theory of analytical functions he has constructed a series named after him and proved several theorems formulated by P. Fermat.

Another, rather new and absolutely original contribution made by Lagrange to development of mechanics was his well-known method of undetermined multipliers, which he had first introduced in statics and then – in dynamics. For the problems of variational calculus with limitations Lagrange formulated the rule of undetermined multipliers in *Analytical Mechanics* in 1788, and in *Theory of Analytical Functions* in 1797 he applied it to finite-dimensional problems.

J.-L. Lagrange considers a sum of arbitrary forces applied to the points of mechanical system, subject to some condition
\[ L = f(x_i) = 0; \quad i = 1, 2, \ldots, n. \quad (4.7) \]

There may be several conditional equations. “They follow from the system nature”. – noted J.-L. Lagrange.

The first variation of the left part of the equation (4.7) \( \delta L \) is also equal to zero, then we obtain a ratio between coordinates variations (virtual displacements).

\[
\delta L = \sum_{i=1}^{3n} \frac{\partial f}{\partial x_i} \delta x_i = 0. \quad (4.8)
\]

The equation (4.8) being multiplied by undetermined multiplier \( \lambda \) and obtained product \( \lambda \delta L \) being added to the left part of the general formula of statics, recorded in projections to Cartesian axes of coordinates, the zero equality of the obtained sum is not disturbed.

\[
\sum_{i=1}^{3n} X_i \delta x_i + \lambda \delta L = 0,
\]

or

\[
\sum_{i=1}^{3n} \left( X_i + \lambda \frac{\partial f}{\partial x_i} \right) \delta x_i = 0,
\]

where additional forces \( \lambda \frac{\partial f}{\partial x_i} \) are added to forces \( X_i \).

J.-L. Lagrange noted as to these forces: “In general the term \( \lambda \delta L \) may be considered as a moment of a certain force \( \lambda \), which causes changes of the function \( L \). Hence it follows that each conditional equation is equivalent to one or several forces applied to the system in the set directions, or try in general to cause changes of values of the set functions... And, on the contrary, these forces can occupy the place of the conditional equations, which proceed from the nature of the set system; thus using these forces, we can consider bodies as absolutely free ones, which are not bound by any constraints. That is the method idea.”

A distinct formulation of the principle of system liberation from constraints, which is the first in the history of mechanics, is contained in the lines of J.-L. Lagrange’s treatise.
The question how the extremum type can be distinguished, and how one can find whether the functional achieves maximum or minimum on the curve obtained from Euler’s equation had not been solved by the beginning of the 19th century. J.-P. Lagrange continued the analogy with differential calculus and studied other variation of the functional. A.M. Legendre and J.-P. Lagrange outlined the paths of studying sufficient conditions of maximum (or minimum) of the functional, but they could not create a general method. K.G.J. Jacobi did it in 1837. But there still existed a number of problems. The major one was insufficient substantiation of the Euler-Lagrange method. “In variational calculus the Euler equations and transversality conditions belong to so-called necessary conditions. They were obtained by the same considerations as in Perron paradox that consists in the following. Let \( N \) be the biggest positive number. Then for \( N \neq 1 \) we have \( N^2 > N \) that contradicts determination of \( N \) as the largest number. Thus \( N=1 \). Then the existence of solution is admitted. This main precondition is created explicitly, and then it is used for searching for solutions, which existence has been postulated. For the classes of problems, where this precondition is fulfilled, such considerations are quite correct. But what kind of class is it? How can we find out whether a concrete problem belongs to this class? The so-called necessary conditions do not answer such question. It is strange that such evident logic error has been left unnoticed for such a long time. The Euler-Lagrange method was first criticized almost 100 years later by K. Weierstrass. Even G.F.B. Riemann made the same unjustified admission in his well-known \textit{Dirichlet’s Principle} [Young, 1974].

K. Weierstrass has considered a set of all extremals of the given problem, i.e. integral curves of the corresponding differential equation, creating preconditions for the theory of the so-called field of extremals constructed by him. Using this theory Weierstrass has found necessary and sufficient conditions for extreems of general form. The difference among them was revealed in the middle of the
century – these two kinds of extremum were called a weak and strong extremum.

Early in the 19th century J.K.F. Gauss and S.-D. Poisson found a necessary condition of the extremum for a double integral, and in 1861 M.V. Ostrogradsky found the same conditions for any ratio.

At the end of the 19th century the results obtained by K. Weierstrass were applied to more general variational problems.

The function called “kinetic potential”, or “function of Lagrange” (Lagrangian)\(^{25}\) – the difference of kinetic and potential energies of the conservative system is widely used in analytical mechanics

\[
L = T - \Pi. \tag{4.9}
\]

Maybe in honor of Lagrange it is signed as L – the first letter of his name.

This function L being introduced the Lagrange equations, called now the second kind equations, look as follows:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \tag{4.10}
\]

Generalized impulses \(p = \frac{\partial L}{\partial \dot{q}_i}\) being introduced, the equation will look:

\[
dp_i = \frac{\partial L}{\partial q_i} \, dt, \quad i = 1, 2, \ldots, m, \tag{4.11}
\]

where \(i\) - the number of the degrees of freedom of the system.

The system of such equations being integrated, generalized coordinates \(q_1, q_2, q_3, \ldots\) will be time functions \(t\) and \(2m\) of arbitrary constants \(\gamma_1, \gamma_2, \ldots, \gamma_{2m}\).

J.-L. Lagrange presents such virtual displacements of the system, which correspond to the variation of arbitrary constants \(\delta \gamma_s\) \((s = 1, 2, \ldots, 2m)\). Then he gives another set of coordinate variations, variations of arbitrary constants \(\Delta \gamma_s\) corresponding to them. The

\(^{25}\) The term kinetic potential was introduced by H. Helmholtz
corresponding changes of generalized coordinates and generalized impulses may be signed by the same symbols $\delta q_i$, $\delta p_i$ and $\Delta q_i$, $\Delta p_i$. According to the rules of variational calculus, i.e. rearrangement of operations $\delta(\Delta)$ and $d$, we can record

$$d \delta p_i = \delta \frac{\partial L}{\partial q_i} dt,$$

$$\Delta dp_i = d \Delta p_i = \Delta \frac{\partial L}{\partial q_i} dt.$$

J.-L. Lagrange multiplies the upper line by $\Delta p_i$, and the lower one by $\delta p_i$, and subtracts the lower line from the upper one. Calculating a sum on all coordinates, he obtains

$$\sum_{i=1}^{m} (\delta p_i \Delta q_i - \delta q_i \Delta p_i) = \text{const.}$$

Thus, as a result of variation of generalized coordinates and generalized impulses the time $t$ is excluded, and a sum in the left part of the equality preserves its value during motion. This result in the theory of variations of arbitrary constants is called the Lagrange lemma [Idelson, 1975, P. 273]. J.-L. Lagrange presents it in the following form:

$$\frac{d}{dt} \sum_{i=1}^{m} \left( \frac{\partial q_i}{\partial \gamma_k} \frac{\partial p_i}{\partial \gamma_s} - \frac{\partial q_i}{\partial \gamma_s} \frac{\partial p_i}{\partial \gamma_k} \right) = 0. \quad (4.12)$$

The sum under the derivative sign is called the Lagrange bracket and is signed as $[\gamma_k, \gamma_s]$. His basic lemma asserts invariability of all such ”brackets” during motion (if we substitute for $q_i$ and $p_i$ their expressions through $t$ and arbitrary constants $\gamma_1, \gamma_2, \ldots, \gamma_{2m}$). Lagrange emphasized himself the importance of the result obtained. He wrote: “Here we have a new very important property of the function $T$, that expresses a live force of the whole system, which can give a general criterion for consideration of the accuracy of solution found with the help of any method”. This formula application is significant for
variation of arbitrary constants in mechanics [Lagrange, 1950, P. 418].

A lot of useful results were further obtained from this formula by S.D. Poisson, R. Hamilton, K.G.J. Jacobi, M. V. Ostrogradsky, J.Liouville, A. Poincare, et al.

Thus the idea of J.-L. Lagrange was realized – to consider arbitrary constants of integration of motion equation not as constants, but as the desired time functions; under their substitution into expressions for $q_i$ and $p_i$ and then into motion equations the latter turn into identity.

R. Hamilton in his work Second Essay on a General Method of Dynamics [Hamilton, 1835] has developed an algorithm following which, when using the so-called Poisson brackets, one can find all $\gamma_k$ depending on time. That is the basis of the modern method of arbitrary constants.

C.G.J. Jacobi has further developed the Hamilton method in the theory of perturbations (celestial mechanics) for integration of perturbation equations for the elements of planet orbits [Jacobi, 1936]. C.G.J. Jacobi has actually developed the ideas of J.-L. Lagrange and S.-D. Poisson concerning variations of the elements of elliptic orbits of celestial bodies, the Lagrange lemma on the properties of Poisson brackets (a variety of Lagrange brackets). Jacobi has revised and estimated the result as that significant for analytical mechanics and for the theory of partial differential equations.

The great influence of the Lagrange ideas is felt in the works of M.V. Ostrogransky. In his work Note on Variation of Arbitrary Constants [Ostrogradsky, 1959-1961, Vol. 2] Ostrogradsky has developed the idea of Lagrange on variation of arbitrary constants to obtain the integrals of conservation of energy, motion of mass centre of the system and planes. In his other work – On Variations of Arbitrary Constants in the Problems of Dynamics [Ostrogradsky, 1968, P. 280-297], he continues developing the Hamilton-Jacobi
method, using the Poisson brackets, when integrating Hamilton canonical equations.

The ideas of J.-L. Lagrange expounded in his lemma (the Lagrange brackets) were further developed by J. Liouville [Liouville, 1849] and A. Poincare [Poircare, 1892].

In mechanics and mathematics Lagrange has made the work on systematization of the obtained results and their substantiation.

Sir Isaac Newton  
(1643 - 1727)

Guillaume François Antoine, marquis de L'Hôpital  
(1661 - 1704)

Gottfried Wilhelm Leibniz  
(1646 – 1716)

Joseph Louis Lagrange  
(1736 - 1813)

J.-L. Lagrange’s outstanding contemporary W. Goete, who understood well the character of this man, wrote: “A mathematician is irreprefacable to such extent, to which he is irreprefacable as a person, as far as he feels the beautiful, inherent in truth; only then his creative work becomes fundamental, pure, clear, inspired, really exquisite” and then “ Lagrange was the ideal man and thus the great. If an ideal man is gifted, he always becomes the good of mankind, carrier of happiness and generosity, whether he is an artist, naturalist, poet or something else”.

J.-L. Lagrange fell ill in spring 1813. Three members of the Institute of France, his friends G. Monge, Lacepede and Chaptal visited him on April 8. J.-L. Lagrange was bedridden more than for a week. They conversed during several hours. J.-L. Lagrange told about most interesting moments of his life. After this visit the scientist fainted away, and in the morning of April 10 he died.
During his last meeting with colleagues J.-L. Lagrange said: “...I feel that I’m dying, my body is growing weak, my mental and physical abilities are fading away: I observe with interest the process of the loss of forces, and I’ll reach the end with no regret, no grief, because the slope is too gentle... I have finished my path, I have become rather known in mathematics. I never nursed a grievance against anybody, never did anything wrong, and I want to finish my path ...” [Delambre, 1867, P. XLIV]. Such was a laconic, restrained, modest appreciation of his life made by J.-L. Lagrange himself.

“Lagrange is the greatest pyramid of mathematical sciences”, – that was the appraisal given by Napoleon Bonaparte to Joseph Louis Lagrange – an outstanding scientist and the most modest mathematician of the 18th century, whom Napoleon made a senator, earl of the Empire and knight commander of a Legion of Honor.

4.2. Legendre transformation. Young inequality. Euler theorem of homogeneous functions

French mathematician Adrien-Marie Legendre (1752-1833) in his works on studying differential equations had revealed one important transformation, which had wonderful properties that determined its use for solving numerous problems of analysis, – the so-called Legendre transformation. This transformation is fundamental in the variational grounds of structural mechanics. Legendre has also established a criterion of extremum existence in variational calculus. And, though the transformation was discovered and used by Euler in 1779, it became widely known after Legendre used it in 1787, and in this connection it was called after A.M. Legendre. It would be historically just to call it the Euler transformation or Euler-Legendre transformation. At the same time the first example of this transformation was found in researches of G. Leibniz. The Young-Fenchel transformation as well as the known Young inequality, double functions by Young, are generalization of this transformation.
Recall that a new function is called the transformations of Legendre function \( f(\xi) \)
\[
H(p) = -f(\xi) + p\xi,
\]
where \( \xi \) is a function of \( p : p = f'(\xi) \).

Legendre transformation is involutive, i.e. its square is equal to identity transformation: if \( f \) under Legendre transformation turns into \( H \), the Legendre transformation of \( H \) will be \( f \) again.

The essence of Legendre transformation consists in a possibility of a double setting of the curve – as a set of points and as an envelope of tangent family.

By definition of Legendre transformation
\[
F(p, \xi) = p\xi - f(\xi) \leq H(p)
\]
for arbitrary \( \xi \) and \( p \). Hence
\[
p\xi \leq f(\xi) + H(p).
\]
(4.13)

The obtained inequality is called Young inequality.

Analogous considerations are also valid for the functions of several independent variables. Assume that \( f(\xi) \) – convex function of the vector variable \( \xi = (\xi_1, \ldots, \xi_n) \), i.e. a quadratic form
\[
\left( \frac{\partial^2 f}{\partial \xi_i^2}, d\xi \right)
\]
is positively definite. Then the function \( H(p) \) is the Legendre transformation of vector variable \( p = (p_1, p_2, \ldots, p_n) \), defined by equalities, analogous to previous ones:
\[
H(p) = F(p, \xi(p)) = \max_{\xi} F(p, \xi),
\]
\[
F(p, \xi) = (p, \xi) - f(\xi),
\]
(4.14)
\[
p = \frac{\partial f}{\partial \xi}.
\]
(4.15)

All previous considerations are also transferred unchanged for this case. The Young inequality also takes place for the functions of several variables
\[
\sum_{i=1}^{n} p_i \xi_i \leq H(p_1, p_2, \ldots, p_n) + f(\xi_1, \xi_2, \ldots, \xi_n),
\]
(4.16)
where \( p_1, p_2, \ldots, p_n \) and \( H(p_1, p_2, \ldots, p_n) \) are defined by formulas (4.15) i (4.14).

Recall, that the function of one or several variables is called homogeneous, if it meets the following condition: at simultaneous multiplying of all arguments of the functions by one and the same arbitrary multiplier the function value is multiplied by a certain degree of this multiplier. I.e. for a homogeneous function \( f(x_1, x_2, \ldots, x_n) \) at all values of \( x_1, x_2, \ldots, x_n \) and arbitrary \( t \) the equality is to be fulfilled

\[
f(tx_1,tx_2,\ldots,tx_n) = t^k f(x_1, x_2, \ldots, x_n).
\]  

(4.17)

Index \( k \) is called \textit{degree of homogeneity} of the function.

In other notations: homogeneous function of \( k \) degree is such a numerical function \( f: \mathbb{R}^n \to \mathbb{R} \), that for arbitrary \( x \in \mathbb{R}^n \) and arbitrary \( t \in \mathbb{R} \) the equality is valid

\[
f(tx) = t^k f(x).
\]

According to Euler’s theorem, if in the expression of a complete differential of homogeneous function \( f(x_1, x_2, \ldots, x_n) \) of \( k \) degree

\[
df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n
\]

differential of each independent variable is changed by this variable we will obtain the function \( f(x_1, x_2, \ldots, x_n) \), multiplied by the degree of homogeneity \( k \):

\[
\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \ldots + \frac{\partial f}{\partial x_n} x_n = k \cdot f(x_1, x_2, \ldots, x_n).
\]  

(4.18)

L. Euler’s theorem on homogeneous functions is widely used in various fields of mathematics, mechanics, physics. Homogeneous functions are also used in economics for modeling “scaled” phenomena.

The equality (4.18) is called \textit{Euler’s formula}. The presented theorem states that this inequality is satisfied by an arbitrary
homogeneous function of $k$ degree, having continuous partial
derivatives. Show the inverse: any function being continuous jointly
with its partial derivatives and satisfying the Euler inequality (4.18),
is necessarily a homogeneous function of $k$ degree.

The given Euler’s theorem confirms, as a matter of fact, the
validity of the Young equality for homogeneous functions. Really, if
we define

$$p_i = \frac{\partial f}{\partial x_i},$$

(4.19)

the presentation of (4.18) will acquire the form

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = f(x_1, x_2, \ldots, x_n) + (k - 1) \cdot f(x_1, x_2, \ldots, x_n).$$

(4.20)

On condition that a determinant, formed from derivatives

$$\frac{\partial^2 f}{\partial x_i \partial x_j},$$

is not equal to zero we shall express from (4.19) $x_i$ through $p_i$,
substitute into second summand of the right part (4.20) and obtain the
function $H(p_1, p_2, \ldots, p_n)$. Thus,

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = f(x_1, x_2, \ldots, x_n) + H(p_1, p_2, \ldots, p_n)$$

− Young equality.

If $f(x_1, x_2, \ldots, x_n)$ – quadratic form, the equality (6.11) takes the form

$$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \ldots + \frac{\partial f}{\partial x_n} x_n = f(x_1, x_2, \ldots, x_n) + f(x_1, x_2, \ldots, x_n),$$

or

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = f(x_1, x_2, \ldots, x_n) + H(p_1, p_2, \ldots, p_n).$$

Record a quadratic form as $f(x) = x^T A x$, where $A$ – matrix of
quadratic form. Since $A$ – a symmetrical matrix, then $A^T = A$. By
Euler’s theorem $p^T x = 2f(x)$, or $p^T x = 2x^T A x$. Hence $p^T = 2x^T A$,
\[ x^T = \frac{1}{2} p^T A^{-1}. \] Then

\[ H(p) = f(x) = \frac{1}{2} p^T A^{-1} \cdot A \cdot \frac{1}{2} A^{-1} \cdot p. \]

Thus

\[ H(p) = \frac{1}{4} p^T A^{-1} p. \]

Note, if the quadratic form looks as

\[ f(x) = \frac{1}{2} x^T A x, \]

then

\[ H(p) = \frac{1}{2} p^T A^{-1} p. \]

Value of the quadratic form \( f(x) \) and its Legendre transformation \( H(p) \) in corresponding points coincide:

\[ f(x) = H(p). \]

### 4.3. Duality for variational principles

It is known that one and the same equation system may be the Euler equation system for different functionals. For example, the equation of analytical mechanics of the systems with the infinite number of degrees of freedom may be obtained proceeding from two different variational principles: Hamilton-Ostrogradsky principle and Hamilton-Poincare principle. In other sections of mechanics the different variational principles were proposed for the same systems of equations: Dirichlet principle and Thompson principle – in mechanics of incompressible liquid and in electostatics, Lagrange principle and Castigliano principle – in the elasticity theory, Pontryagin principle of maximum – in variational constrained problems, etc. It has later become clear that all such principles are based on a simple general idea – the idea of duality.

For the simplest problem on the central tension of the bar with rigidity \( EF \) by force \( N \) at the right end we will obtain the following posing of dual problems according to Lagrange.

**Direct problem**
\[
\Pi^D(u) = \frac{1}{2} \int_a^b \! EF(u')^2 \, dx - \int_a^b \! q_s \, u \, dx - N_b u_b \rightarrow \min.
\]

**Dual problem**
\[
\Pi^C(N) = -\frac{1}{2} \int_a^b \! \frac{N^2}{EF} \, dx \rightarrow \max.
\]
\[ u_a = 0. \]

A functional has minimum equal to
\[ \Pi^L_{\text{min}}(u) = -\frac{N_b^2 l}{2EF}. \]

\[ \frac{dN}{dx} + q_x = 0; \quad N = N_b. \]

A functional has maximum equal to
\[ \Pi^C_{\text{max}}(N) = -\frac{N_b^2 l}{2EF}. \]

Functions \( \Pi^L(u) = \frac{EF}{2l}u_b^2 \) and \( \Pi^C(N) = -\frac{1}{2} \frac{N_b^2 l}{EF} \) are dual according to Young, and Young inequality may be recorded in the form of equality:

\[ \frac{EF}{2l}u_b^2 + \frac{1}{2} \frac{N_b^2 l}{EF} = N_b u_b, \]

which is a principle of energy saving, i.e. equality of work of the internal and external forces.

Legendre transformation in the case of the function of forces and displacements of the finite number of variables

\[ N^T = \{N_1, N_2, \ldots, N_n\}, \quad \Delta^T = \{\Delta_1, \Delta_2, \ldots, \Delta_n\} \]

is constructed as follows [Bazhenov, 2014]:

Potential energy of elastic deformation

\[ U(\Delta) = \frac{1}{2} \Delta^T K \Delta. \]

Complementary potential energy

\[ U^{\text{comp}}(N) = \frac{1}{2} N^T B N. \]

Legendre transformation

\[ \frac{1}{2} \Delta^T K \Delta + \frac{1}{2} N^T B N = N^T \Delta. \]

In so doing the conditions of equilibrium, jointness of deformations, boundary conditions should be fulfilled.

The condition which turns Young inequality into equality

\[ N = K \Delta, \quad \text{or} \quad \Delta = B N, \]

Thereat the matrices \( K \) and \( B \), which are, respectively, the
rigidity and compliance matrices are mutually inverse ones \( \mathbf{K} \mathbf{D} = \mathbf{E} \). These matrices are the matrices of second derivatives (Hesse matrices) of potential energy of elastic deformation and complementary potential energy, their coefficients are equal to

\[
k_{ij} = \frac{\partial^2 U(\Delta)}{\partial \Delta_i \partial \Delta_j}; \quad \delta_{ij} = \frac{\partial^2 U^{\text{comp}}(\mathbf{N})}{\partial \mathbf{N}_i \partial \mathbf{N}_j}.
\]

According to Donkin theorem, if two dual (by Young) functions of potential energy \( U(\Delta) \) and \( U^{\text{comp}}(\mathbf{N}) \) depend on the same parameter or a group of parameters, which are active, i.e. take no part in Legendre transformation \( (\eta) \), then the dependence takes place:

\[
\frac{\partial U(\Delta)}{\partial \eta} = -\frac{\partial U^{\text{comp}}(\mathbf{N})}{\partial \eta},
\]

\[
U(\Delta) = \frac{1}{2} k \Delta^2, \quad U^{\text{comp}}(\mathbf{N}) = \frac{1}{2} \frac{N^2}{k},
\]

\[
\frac{\partial U(\Delta)}{\partial k} = \frac{1}{2} \Delta^2, \quad \frac{\partial U^{\text{comp}}(\mathbf{N})}{\partial k} = -\frac{1}{2} \frac{N^2}{k^2} = -\frac{1}{2} \Delta^2.
\]

The corresponding extremal problems in the Legendre transformation give dual (by Legendre) posings of extenal problems.

**Direct problem**

\[
\left\{ \frac{1}{2} \Delta^T \mathbf{K} \Delta - \mathbf{N}^T \Delta \right\} \rightarrow \min,
\]

under condition \( \Delta = \bar{\Delta} \).

**Dual problem**

\[
\left\{ -\frac{1}{2} \mathbf{N}^T \mathbf{B} \mathbf{N} + \mathbf{N}^T \bar{\Delta} \right\} \rightarrow \max,
\]

under condition \( \mathbf{N} = \bar{\mathbf{N}} \).

Note that the adjoint function to the function \( f : X \rightarrow R \), defined on the vector space \( X \), which is in duality (relative to bilinear form \( \langle x, y \rangle \)) with the vector space \( Y \) is called the function on \( Y \), set by the ratios [Mathematical Encyclopedia. Vol. 5, 1985, pp. 82-83]

\[
f^*(y) = \sup_{x \in X} \left( \langle x, y \rangle - f(x) \right).
\]  

(4.21)
For the function set on $Y$, the adjoint function is defined analogously.

If $f(x)$ — a smooth function, growing on the infinity more quickly than the linear one, then $f^*(y)$ is Legendre transformation of the function $f(x)$.

The definition for unidimensional strictly convex functions equipotent to (4.21), was given by V. Young in other terms. V. Young has defined the function adjoint to the function

$$ f(x) = \int_0^x \varphi(t) \, dt , $$

where $\varphi(t)$ is continuous and strictly grows, by the ratio

$$ f^*(y) = \int_0^y \psi(t) \, dt , $$

where $\psi(t)$ — function inverse to $\varphi(t)$.

The **Young** inequality is fulfilled for a convex function and that adjoint to it

$$ \langle x, y \rangle \leq f(x) + f^*(y) . $$

232
The adjoint function is the convex closed function. Conjugation operator $*: f \rightarrow f^*$ reflects unambiguously a set of convex eigenfunctions on $X$ on a set of convex eigenfunctions on $Y$.

References


Vupper, Yu.F. (1875), *Semeistvo matematikov Bernulli* [Bernoulli family of mathematicians], Moscow, Russia.


Gantmakher, F.R. (1966), *Lektsii po analiticheskoi mekhanike* [Lecture in analytical mechanics], Moscow, USSR.
Gelfand, I.M. and Fomin, S.V. (1961), Variatsionnoye ischisleniye [Variational calculus], Fizmatgiz, Moscow, USSR.

Gerts, G. (1959), Printsipy mekhaniki, izlozhennyie v novoi svyazi [Principles of mechanics stated in new respect], AN SSSR, Moscow, USSR.

Gnedenko, B.V. (1952), Mikhail Vasilievich Ostrogradsky, GITTL, Moscow, USSR.

Gnedenko, B.V. and Pogrebysskiy, I.B. (1963), Mikhail Vasilievich Ostrogradsky, Izdatelstvo AN SSSR, Moscow, USSR.

Goldstein, G. (1957), Klassicheskaya mekhanika [Classical mechanics], Gostekhizdat, Moscow, USSR.

Grigoryan, A.T., (1961), Ocherki istorii mekhaniki v Rossii [Essays in the history of mechanics in Russia], Izdatelstvo AN SSSR, Moscow, USSR.

Grigoryan, A.T., (1964), Mikhail Vasilievich Ostrogradsky, Izdatelstvo AN SSSR, Moscow, USSR.

Grigoryan, A.T., (1974), Mekhanika ot antichnosti do nashikh dnei [Mechanics from Antiquity to our days], Nauka, Moscow, USSR.

Grigoryan, A.T. and Kovalev, B.D. (1981), Daniil Bernulli [Daniel Bernoulli], Nauka, Moscow, USSR.

Dorofeyeva, A.V. (1961), Razvitiye variatsionnogo ischisleniya kak ischisleniya variatsii. Istoriko-matem issledovanie. [Development of variation calculus as a calculus of variations. Historico-mathem. Research, Iss.XIV], Fizmatgiz, Moscow, USSR.

Duhem, P (1903), Razvitiye mekhaniki [Development of mechanics], Russia.

Idelson, N.I. (1975), Etyudy po istorii nebesnoy mekhaniki [Studies in the history of celestial mechanics], Nauka, Moscow. USSR.

Bogolyubov, A.N. and Shtokalo, I.Z. , Eds (1987), Istoriya mekhaniki v Rossii [The history of mechanics in Russia], Naukova dumka, Kiev, USSR.

Grigoryan, A.T. and Pogrebyssky, I.B., Eds (1971), Istoriya mekhaniki s drevneishikh vremyon do kontsa XVIII veka [The history of mechanics from the ancient times to the end of the 18th century], Nauka, Moscow, USSR

Grigoryan, A.T. and Pogrebyssky, I.B., Eds (1972), Istoriya mekhaniki s XVIII veka do serediny XX veka [The history of mechanics from the end of the 18th century to the middle of the 20th century], Nauka, Moscow, USSR

234
Keppler, I. (1935), *Novaya stereometriya vinnykh bochek* [New stereometry of wine tuns], ONTI-GTTI, Moscow-Leningrad, USSR.

Klein, F. (1937), *Lektii o razvitii matematiki v XIX stoletii* [Lecture on development of mathematics in the 19th century], ONTI, Moscow-Lenngrad, USSR.


Kosmodemiansky, A.A. (1982), *Ocherki po istorii mekhaniki* [Essays in the history of mechanics], Nauka, Moscow, USSR.

Kotek, V.V. (1961), *Leonard Euler* [Leonard Euler], Uchpedgiz, Moscow, USSR

Krylov, A.N. (1935), *Leonard Euler. 1707-1783. Sbornik statei i materialov k 150-letiyu so dnya smeri* [Leonard Euler. 1707-1783. Collected papers and materials on the 150 anniversary of his death], Izdatelstvo akademii nauk SSSR, Moscow-Leningrad, USSR.


Krylov, A.N. (1943), *Nyuton i yego znacheniiye v mirovoi nauke (1643-1943)* [Newton and his significance in the world science (1643-1943)], Izdatelstvo akademii nauk SSSR, Moscow-Leningrad, USSR.

Kudryavtsev, P.S. (1943), *Isaak Nyuton, 1643-1943. K 300-letiyu so dnya rozhdeniya* [Isaak Newton, (1643-1943). On the 300th anniversary of his birth], Uchpedgiz, Moscow, USSR.


Kudryavtsev, P.S. (1955), *Isaak Nyuton* [Isaak Newton], Uchpedgiz, Moscow, USSR.

Kudryavtsev, P.S. (1974), *Kurs istorii fiziki* [Course of the history of physics], Prosveshchenie, Moscow, USSR.

235


Lagrange, J. (1950), Analiticheskaya mekhanika, Tom 1 [Analytical mechanics, Vol. 1], GITTL, Moscow-Leningrad, USSR.

Lagrange, J. (1950), Analiticheskaya mekhanika, Tom 2 [Analytical mechanics, Vol. 2], GITTL, Moscow-Leningrad, USSR.

Levi-Civita, T. and Amaldi, U. (1934), Kurs teoreticheskoi mekhaniki. T.1, Ch. 1 [Course of theoretical mechanics. Vol.1, P. 1], ONTI, Moscow, USSR.

Levi-Civita, T. and Amaldi, U. (1962), Kurs teoreticheskoi mekhaniki. T.1, Ch. 2 [Course of theoretical mechanics. Vol.1, P. 2], Moscow, USSR.

Leibniz, H.V. (1982), Sochineniya v 4 tomakh. T. 1, [Collected works in 4 volumes, Vol. 1], Moscow, USSR.


Mach, E. (1909), Mekhanika [Mechanics], St Petersburf, Russia.


Mikhlin, S.G. (1970), Variatsionnye metody v matematicheskoi fizike [Variational methods in mathematical physics], Nauka, Moscow, USSR.

Newton, I. (1936), *Matematicheskie nachala naturalnoi filosofii* [Mathematical principles of natural philosophy], Transl. from Latin A.N. Krylov, Izd. AN SSSR, Moscow-Leningrad, USSR.


Ostrogradsky, M.V. (1959-1961), *Polnoye sobraniye trudov v dvuh tomakh* [Complete collected works in two volumes], Izd. AN Ukr.SSR, Kiev, USSR.

Ostrogradsky, M.V. (1968), *Izbrannye trudy* [Selected works], AN SSSR, Moscow, USSR.

Polak, L.S. (editor) (1959), *Variational printsipy mekhaniki. Sbornik statei klassifik nauki* [Variational principles of mechanics. Collected papers of sciecee classics], Fizmatgiz, Moscow, USSR.

Polak, L.S. (editor) (2010), *Variatsionnye printsipy mekhaniki.* [Variational principles of mechanics], ed. 2, revised, Knizhnyi dom LIBORKOM, Moscow, Russia.


Poincare, A. (1910), *Nauka i method* [Science and method], St Petersburg, Russia.


Poincare, A. (1983), *O nauke* [About science]. Nauka, Moscow, USSR.

Tyulina, I.A. (1979), *Istoriya i metodologiya mekhaniki* [The history and methodology of mechanics], Izd. Moskovskogo universiteta, Moscow, USSR.


Collected papers and materials on the 150th anniversary of death], Izdatelstvo AN SSSR, Moscow-Leningrad, USSR.

Schmutzer, E. and Schutz, V. (1987), *Gallileo Gallilei* [Gallileo Gallilei], Mir, Moscow, USSR.

Euler, L. (1934), *Metod nakhozhdeniya krivykh linii, obladayushchikh svoistvami maksimuma libo minimuma, ili resheniya izoperimetriceskoi zadachi, vzayoty v samom shirokom smysle* [Method of finding curves possessing properties of maximum or minimum, or solution of the isoperimetric problem, taken in the broadest sense], GTTI, Moscow-Leningrad, USSR.


Euler, L. (1958), *Integralnoye ischisleniye, v. 3* [Integral calculus, t. 3], Fizmatgiz, Moscow, USSR.

Elscholtz, L.E. (1958), *Variatsionnyie ischisleniye* [Variational calculus], GITTL, Moscow, USSR.

Elscholtz, L.E. (1965), *Differentsialnyie uravneniya and variatsionnoie ischisleniye* [Differential equations and variational calculus], Nauka, Moscow, USSR.


Jacobi, K. (1936), *Lektii po dinamike* [Lectures in dynamics], ONTI, Moscow-Leningrad, USSR.

Young, L. (1974), *Lektii po variatsionnomu ischisleniyu i teorii optimalnogo upravleniya* [Lectures in variational calculus and theory of optimal control], Mir, Moscow, USSR.

D’Alembert, J.R. (1743), Traité de dynamique, dans lequel les lois de l'équilibre & du mouvement des corps sont réduites au plus petit nombre possible, David L'aîné, Paris, France.

D'ALEMBERT, J.R. (1867), Notice sur la vie et les ouvrages de Lagrange (Œuvres de Lagrange, V.1), Paris, France.


De l'Hospital, G.F. (1696), Analyse des infiniment petits, pour l'intelligence des lignes courbes, Paris, France.

Euler, L. (1744), Methodus inveniendi lineas curvas maximis minimive proprietate gaudentes. Marc Michel Bousquet, France.


Fuss, P.H. (1843), Correspondence mathematique et physique, page 26, volume II, St.Peterburg, Russia.

Galileo Galilei (1638), Discorsi e dimostrazioni matematiche intorno a due nuove scienze. In Leida.


Helmholtz, H. (1902), Dynamik continuerlich verbreiteten Massen, Verlag von Johann Ambrosius Barth, Leipzig, Germany.

Hertz, H. (1880), Über die Induktion rotierender Kugeln, Berlin, Germany.


Isis. XX. (1933), Brugge, Belgique.

Lagrange, J.L. (1788), Mécanique analytique, 1re éd, Paris, France.

Leibniz, G.W. (1684), Demonstrationes novae de Resistentis solidorum, Acte Eruditorum Lipsiae, pp. 319-325.


Mach, E. (1883), Die Mechanik in ihrer Entwicklung, historisch-critisch dargestellt, Leipzig, Germany.


Poincare, H. (1892), Les methodes nouvelles de la Mécanique celeste, t. 1, Paris, France.

Spiess, O. (1948), Die Mathematiker Bernoulli, Basel, Switzerland.

Truesdell, C. (1966), Six Lectures on modern natural philosophy, Springer-Verlag, New York, USA.


Varignon, P. (1725), Nouvelle mécanique, 2, Paris, France.
Essay 5

VARIATIONAL AND EXTREMAL PRINCIPLES OF MECHANICS. THE HISTORY OF DEVELOPMENT
Statics is the science about equilibrium of forces.

J. L. Lagrange

As is well known the principle of virtual velocities transforms all statics into a mathematical assignment, and by D'Alembert's principle for dynamics, the latter is again reduced to statics... It follows that neither fundamental principle of equilibrium and motion can differ significantly from the two above principles, and that whatever this principle be, it can always be considered as a more or less direct conclusion from the above ones.

J.K.F. Gauss
Introduction

Each variational principle asserts that such a state, a process is actually realized for a certain class of problems (if the problem conditions are prescribed) from all imaginary states, processes, etc., in a sense consistent with these conditions; this state, process gives a certain functional, characteristic of this principle, a stationary value. Sometimes the question is not of the stationary, but of the extreme value that is not always equivalent. Variational principle is equivalent to a certain system of differential equations. So variational principle is characterized by a class of problems, conditions of compatibility of processes that are compared to conditions of the problem and to a functional defined for these processes, which must take a stationary or extreme value.

As to terminology it should be noted that, strictly speaking, the term variational principle should be used only in respect of the systems with the infinite number of degrees of freedom, preserving the term extreme principle for the systems with the finite number of degrees of freedom. But in most cases all the extreme principles are called variational ones regardless of the finite or infinite number of degrees of freedom of the systems to which those principles are applied. The more so that the principle nature does not change depending on the system to which it is applied [Goldenblatt, 1957].

Generally there are two main groups of variational principles. The above definition is most suitable for one of them, which begins with the principles of Fermat, Maupertuis, Lagrange, Castigliano, Hamilton et al., and deals with the tasks of physics, mechanics and other disciplines. It refers to the study of objective processes. This principle can be deduced from a local theory of the process, with its Euler’s differential equations for the functional characterizing this principle. In this case, it is more about the stationary functional value, because this value extremality is insignificant for the local properties. However, there may be classes of problems for which extremality
carries additional information, for example on the principle validity. It should be noted that, if the variational principle is equivalent to the differential equation, this principle is often useful due to its greater versatility and ease of the use of numerical methods.

The second major group of variational principles deals with the problems of mathematical economics, optimal control. The point is in the development of a certain strategy in set conditions that would ensure maximum benefit in these conditions. The significance of this benefit, the purpose function is the functional, and, as a rule, this functional should be maximized. The corresponding mathematical theory was created in the mid-fifties of the past century and called optimal control theory. Optimal control theory is a synthesis of ideas and research methods of the classical variational calculus and of modern ones.

Like mathematical analysis built in the 18th century by analogy with the basic algebraic operations, the calculus of variations was also built by analogy with well-known operations of analysis and was assumed as the basis of the latest mathematical methods of the optimal control theory. As the famous American mathematician Lawrence Young [Young, 1974] emphasized: "Now it is seen that the Lagrange problem is not substantially different from the optimal control problem: only the latter is a more modern formulation of the former. Small noticeable differences are indicated sometimes, but they are really quite insignificant."

The rapid development of the theory of optimal control and its applications to practical problems, the well-known Pontryagin’s maximum principle – all that, on the one hand, stimulated interest in variational problems, but on the other, led to fairly widespread attitude to classic variational calculus, as certain anachronism. In fact, as L. Young [Young, 1974] defines, it is not so. It is not only that the calculus of variations by L. Young is a "chronicle of mathematical concepts. And as progress in mathematics is largely associated with the emergence of new concepts, the calculus of variations is the current direction for further research; in this respect
no branch of mathematics does not compare with variational calculus”.

Research of the partial problems of variational calculus, or as we say partial variational problems, started an extremely long time ago. This is because in many cases people are satisfied only with the best possible options. So we come to optimization problems, i.e. the problems of finding the maximum or minimum. Some of these problems were resolved using basic tools of mathematical analysis, but in most cases their solutions require more complicated methods.

In fact, these tasks require not only new methods, but, more importantly, new concepts. Never start solving such problems with bare hands; you must first take care of good equipment. Required concepts evolved slowly, and though they now permeate all of modern mathematics, most people are not even aware that these notions have emerged from the calculus of variations. This is partly due to the fact that the calculus of variations has preserved the old-fashioned terminology.

The very name of the subject is purely historical; it relates to a method dating back to L. Euler and based on the so-called variations. This method once played an important role in calculus of variations, but now it is of secondary importance. Calculus of variations was actually one of the sections of functional analysis and it takes the same place as the theory of maxima and minima in the ordinary analysis.

If so, it is not surprising that the notions that, as is customary to think, belong to functional analysis or derived from it, actually first emerged (perhaps in a less elegant form) in the calculus of variations. Thus, a number of basic tools of modern mathematics, such as L.Schwartz generalized functions ("distribution"), Young's inequality as well as convex figures and their polars can be found as germs in the classical methods of the calculus of variations.

This means that the calculus of variations appears not only as a branch of mathematics, but also as a chronicle of mathematical concepts. And since progress in mathematics is largely linked with
the emergence of new concepts, the subject of study here gives us a desperately needed guide to further research in this respect; no field of mathematics can be compared to the calculus of variations.

As for direct connection of variational calculus with other branches of mathematics, the reader will soon realize the important role of fundamental questions of pure logic in variational calculus and role of homology in the Morse theory that unites it both with algebra and topology. Topologists also know important applications of the Morse theory or methods based on the ideas similar to their subject. However, the calculus of variations is mainly used in analysis and geometry or in connection with obtaining certain inequalities and estimates (as in the theory of partial differential equations) or in the development of qualitatively new methods and concepts (as it was with Riemann, who applied the "Dirichlet principle of minimum" to conformal mappings and potential theory; this principle was subsequently substantiated in the famous work of D. Hilbert).

It might be concluded from the above that the calculus of variations is a rather "pure" science, which has a remote relationship to applied mathematics. In fact, it is quite the contrary. Not only specific problems of variational calculus play a major role in everyday life and such fields as economics, engineering, etc. (humans try to do their best in the framework of the existing possibilities), but the theory itself owes all its development to optics requirements and later the requirements of space sciences and other related issues.

To the purpose, some of the famous "D. Hilbert problems" (twenty, in particular) are devoted to calculus of variations [Young, 1974].

There is even a view that the history of mechanics and physics is the history of attempts to explain everything that happens in the world around us, with a small quantity of universal laws and general principles. The most successful and fruitful efforts are associated with the idea that the phenomena that we observe have some extreme properties and general principles are of variational character, i.e. they
say that in some real-world processes some magnitudes reach their maximum or minimum value [Berdychevskyy, 2005].

Variational principles of mechanics are of great theoretical and applied value, since:

- show overall energy nature of the phenomena under study;
- allow obtaining automatically the equilibrium (motion) equations, static boundary conditions or strain compatibility equations and kinematic boundary conditions, which sometimes cannot be strictly obtained by other methods;
- the functional subject to variation has a defined physical meaning and is invariant in respect of coordinate transformation. That is, if the variational principle is formulated in one coordinate system, you can get governing equations in the other coordinate system, and then make a variation. For example, if the variational principle was formulated in a rectangular Cartesian coordinate system, the governing equations in cylindrical or spherical coordinate system may be obtained through these procedures. This property makes variational methods extremely effective in the structure study;
- variational formulation is convenient for common mathematical procedures such as the given problem transformation into equivalent one, which solution is simpler than that of the original problem. In variational formulation with additional conditions this transformation is performed using Lagrange multipliers. So you can get a family of variational principles equivalent to each other. These transformations, in particular, include the well-known Friedrichs transformation;
- it is significant that the variational principles sometimes lead to formulas for the upper and lower bounds of the problem. Usually when assessing the accuracy of thus obtained solutions one should exercise caution;
- make it possible to find solutions bypassing formulation and solution of differential equations, using the so-called direct methods of the calculus of variations, being the most convenient for implementation in modern numerical procedures.
Variational principles acquired special significance only now in connection with the development of finite element method, which is widespread since the pioneer work by J. Turner [Turner et al. 1956], works by R. Clough [Clough, 1960], J. Argyris [Argyris, 1954, 1955, 1957], A. Zienkiewicz [Zienkiewicz & Cheung, 1967] and others. It has repeatedly been proved that the variational principles are an effective tool in mathematical formulation of the finite element method and that, on the contrary, accelerated development of the finite element method stimulated the improvement of variational principles.

In continuum mechanics, there are many different and seemingly unrelated variational principles. Variational formulations of the problems can be presented either in the form of integral identities that are considered in mechanics as variational principles and are called variational equations or as stationary requirements of appropriate functionals that are considered in mechanics as variational principles. The variational equation for continuous medium is assumed as a basis of systematization. The variational principles may be of different character depending on the fact which functions vary, and which are used as constraints dependences. Further transformations are based on the idea of duality. Direct minimization problem of Lagrange functional is associated with the dual maximization problem of the Castigliano functional with opposite sign. Thereat the extreme values of the corresponding functionals coincide. Mixed functionals are derived following such a model.

5.1. Principles of possible displacements

The principle of possible displacements is the differential variational principle of classical mechanics, which expresses the most general terms of equilibrium of mechanical systems with ideal constraints.

According to this principle the mechanical system is in equilibrium in a certain position when, and only when the sum of elementary works of set active forces at any possible displacement,
which withdraw the system of the considered position is zero or less than zero:

$$\sum_{v=1}^{N} \mathbf{F}_v \cdot \delta \mathbf{r}_v \leq 0$$  \hspace{1cm} (5.1)$$

at any time.

Elementary (infinitesimal) displacements $\delta \mathbf{r}_v$ of the system points admitted at a given time by constraints imposed on the system are called possible system displacements. Projections of possible displacements to Cartesian axis coordinates are called variations of Cartesian coordinates [Mathematical Encyclopedia, 1977].

If the constraints are bilateral, the displacements may be backward, and the equality sign should be taken in condition (5.1); if the constraints are unilateral, there are irreversible displacements among the possible ones. When the system moves under the effect of active forces the constraints act on the system points with some reaction forces $\mathbf{R}_v$ (passive forces); their determining provides the entirely accounted mechanical action of constraints on the system (in the sense that the constraints may be replaced by the reactions caused by them) (release axiom). The constraints are considered perfect if the sum of elementary works of their reactions $\sum \mathbf{R}_v \cdot \delta \mathbf{r}_v \geq 0$, the equality sign taking place for the backward possible displacements and equality signs or those above zero – for irreversible movement. System equilibrium positions are such positions $\mathbf{r}_v = \mathbf{r}_v(t_0)$, where the system will remain all the time, if it is placed in these positions with zero initial velocities $\mathbf{v}_v(t_0) = 0$; it is assumed that the equations of constraints are satisfied at any $t$ by the values $\mathbf{r}_v = \mathbf{r}_v(t_0)$ and $\mathbf{v}_v = 0$. Active forces in a general case are considered as set functions $F_v(t, r_{\mu}, v_{\mu}) \in C^1$, and in condition (5.1) they should be considered $F_v(t, r_{\mu}(t_0), 0)$.

The condition (5.1) contains all the equations and laws of equilibrium of the system with ideal constraints, so we can say that all the statics is reduced to one general formula (5.1).
It should be noted that the principle of virtual displacements is formulated in the course of *Analytical Mechanics*. However, virtual displacement are treated as the infinitesimal ones allowed by constraints, under the "frozen time" rather than under fixed one, as in the case of possible displacements. That is virtual displacements differ from the possible displacements only when the constraints are rheonomous (obviously depending on time).

For example, if \( l \) holonomous rheonomous constraints are imposed on the system:

\[ f_\alpha(r,t)=0, \quad \alpha=1,l, \]

possible displacements \( \Delta r \) are those satisfying the equality

\[ \sum_{\alpha=1}^{N} \frac{\partial f_\alpha}{\partial r} \Delta r + \frac{\partial f_\alpha}{\partial t} \Delta t = 0, \quad \alpha=1,l, \]

and virtual \( \delta r \):

\[ \sum_{\alpha=1}^{N} \frac{\partial f_\alpha}{\partial r} \delta r = 0, \quad \alpha=1,l. \]

Virtual displacements, generally speaking, are not related to the system motion process – they are introduced only to identify the existing balance of forces in the system and to obtain the equilibrium conditions. Smallness of displacements is required to consider the reactions of ideal constraints unchanged.

Note that possible displacements do not enter in formulations of variational principles and methods by themselves, but as multipliers among the expressions of possible work. A possible work of force \( F \), which corresponds to a possible displacement \( \Delta r \) is called elementary work

\[ F \cdot \Delta r = F \Delta r \cos(F, \Delta r). \quad (5.2) \]

The two following rules may be applied to this possible work as well as to usual one [Appel, 1960]

1. For the same possible displacement \( \Delta r \) of a point \( M \) the resultant system work of several forces applied to this point is equal to the sum of work of forces within the system.
If possible displacement $\Delta \mathbf{r}$ is a geometrical sum of several displacements, the work of one and the same force on displacement $\Delta \mathbf{r}$ equals the sum of works of these forces on the individual components of displacement.

If we sign by $\delta t$ infinitesimal time interval, during which a possible displacement $\Delta \mathbf{r}$ is performed, then the vector $\mathbf{V}$, equal to $\frac{\Delta \mathbf{r}}{\delta t}$ and directed along $\Delta \mathbf{r}$, is called possible velocity, given to point $M$. If $\Delta \mathbf{r}$ is substituted by expression $\mathbf{V} \delta t$, possible work may be recorded as follows

$$\mathbf{F} \cdot \mathbf{V} \delta t = F V \cos(\mathbf{F}, \mathbf{V}) \delta t.$$  \hspace{1cm} (5.3)

Since an angle between the force $\mathbf{F}$ and vector $\mathbf{V}$ is equal to the angle between this force and displacement $\Delta \mathbf{r}$.

Analytically, in a rectangular system, if force projection is denoted by $X, Y, Z$, and the displacement projection by $\delta x, \delta y, \delta z$, the possible work may be represented as

$$X \delta x + Y \delta y + Z \delta z.$$  

Nowadays a possible work is usually taken in the form (5.2), though it was often taken in the form (5.3). However, the form (5.3) is used and if possible displacements of several different points are considered, it is considered that the interval $\delta t$ has one and the same value for all these points.

### 5.2. Fundamentals of Statics

It is believed that the principle of possible displacements (velocities) historically was the first variational principle. Its idea in the original form was contained in *Physics* of the famous ancient Greek philosopher Aristotle (384-322 BC). And Aristotle, as Euler notes [Euler, 1753], probably borrowed this idea from his predecessors.

It is relevant to recall the first integral variational principles. Probably classical isoperimetric problem is the oldest of the known extreme problems. It is difficult to say when a thought was first
suggested, which concerned the largest "capacity" of the circle and sphere among all closed curves of equal length or surfaces of the same area. One of the last students of the Athens School of Platonists Simplicius of Cilicia (VI cent. BC), who had composed the great commentary on the works of Aristotle not long before the final collapse of ancient civilization, wrote: "it was proved even before Aristotle, because he used it as well-known, and then to a higher extent – by Archimedes and Zenodorus that a circle is the most capacious among isoperimetric figures and a sphere among isopiphic ones ". These words defined a setting of such extreme problems: among plane closed curves with a given length, find the curve which covers the largest area, and among spatial closed surfaces with a given area, find a surface that encloses the largest volume. For the philosopher-Platonist such formulation of the problem is natural and associated with a search of ideal forms. No wonder the circle and the sphere were the ancient symbols of geometric perfection.

Prosaic motivation of isoperimetric and related problems can be found in the somewhat naive, but quite expressive form in the legend of Dido. Recall it by Aeneid by the Roman poet Virgil. Phoenician Princess Dido with a small detachment of residents of Tyre, fleeing the persecution of the tyrant – Dido's brother, left her hometown and went on ships westward along the coast of the Mediterranean Sea. Choosing the African coast convenient (present Gulf of Tunis), Dido and her companions decided to found a settlement. Apparently, the locals did not like this idea, but Dido managed to persuade their leader Yarba and he inadvertently agreed to cede a piece of land, "which can be surrounded by bovine skins." Yarba did not realize cunning and deceit of the Phoenician. Dido had cut the skin into thin strips, bound them into one long strap and surrounded great area, where laid the city of Carthage (Phoenician Kartadasht – "new city"). In memory of this story the citadel of Carthage was named Beers (skin – in the language of Punicians, as the Romans called the inhabitants of Carthage). The legend refers all these events to 825 (or 814) BC.
The same classical isoperimetric problem arises in this situation: to specify the optimum form of land plot, which at a given perimeter length $L$ has the largest area $S$.

Solving the isoperimetric problem is given by the following statement: if a rectifiable curve of $L$ length limits the plane figure with the area $S$, then

$$L^2 \geq 4\pi S$$  \hspace{1cm} (5.4)

and equality holds if and only if the curve is a circle.

Other problem formulations could be obtained if, as is natural to conceive, Dido wanted to have access to the sea. Unlike classical isoperymetric problem, these problems are often called the Dido problems.

We find the principle of possible displacements in embryo state in the mechanics of Leonardo da Vinci – one of the few universal geniuses in the history of mankind.

The first knowledge in mechanics before the 6th century BC concerned (using modern terminology) such sections as hydraulics, structural mechanics, statics, dynamics and celestial mechanics. The simplest devices or so-called "simple machines": levers, inclined planes were used in the construction process since ancient times. Pictures of cup balance on Egyptian papyrus is the best evidence of acquaintance of ancient Egyptians with equal-arm lever. Egyptian well "crane" (shaduf) – a witness of acquaintance with unequal arm
lever. Ancient Greeks also knew these machines, paddles and helm of Greek galleys being a proof.

The idea of the principle of virtual displacements was expressed by S. Stevin, when he considered equilibrium conditions of the block. The next step was made by Galileo, when considering the balance of the body, lying on an inclined plane. He expressed the famous "golden rule" of mechanics, according to which "what is won in effect, the loss in speed (displacement)"\(^{26}\).

In some simplest cases, the principle justification is not difficult. Let two loads \(F_1\) and \(F_2\) be balanced at the ends of weightless lever or block (frictionless). Then signifying through \(F'_1\) and \(F'_2\) tangential (to the possible trajectories) components of these forces, and through, \(\delta l_1\) and \(\delta l_2\) – the value of respective possible elementary displacements, by virtue of equality (5.1) with the accuracy to a sign we shall have

\[ F'_1 \delta l_1 = F'_2 \delta l_2, \]

That is

\[ \frac{\delta l_1}{\delta l_2} = \frac{F'_2}{F'_1}, \]

gain in force is compensated for by a loss in displacement and vice versa.

But S. Stevin and Galileo solved some problems of uncommon character. In 1717 Johann Bernoulli had formulated the principle of virtual displacements for any given system of applied forces, although he did not provide the proof of this principle. J.-L. Lagrange in his *Analytical Mechanics* formulated the principle, called it "the principle of virtual velocities" and gave its proof (although not strict) by means of the system of loads, suspended on wires thrown over blocks. Andre-Marie Ampere (1775 1836) substantiated the principle of virtual displacements and introduced a postulate of ideal constraints. M.V. Ostrogradsky has extended the principle of virtual displacements to continuous media.

\(^{26}\) Galileo attributed to Aristotle justification of the "golden rule of mechanics."
displacements to the systems with non-stationary and unilateral constraints.

Evangelista Torricelli (1608-1647) was the Italian physicist and mathematician. In 1641 he was invited by Castelli to Arcetri, where he became a student and secretary of already old, sick and blind Galileo, helping him prepare manuscripts for publication. Three months later, Galileo died, Torricelli inherited his post of the mathematician of Duke of Toscana and became his successor at the Department of Mathematics and Philosophy of the University of Florence. Torricelli was a great geometrician, but he is also known to posterity by his discoveries in mechanics, including the discovery of atmospheric pressure, the invention of barometer and formula of leakage of heavy fluid from the vessel through the opening. This formula is contained in the memoirs «De Motu Gravium Naturaliter Descen dentium» (On the Motion of Heavy Bodies that Fall Naturally).

Derive the equilibrium conditions of an arbitrary constrained system of solids which are under the influence of gravity. Denote by $M$ the sum of the masses of all bodies and by $z_c$ – the vertical coordinate of the center of gravity of the system of bodies (consider the whole $z$ axis is directed vertically downward). Then, according to equality (5.1), we get

$$Mg \delta z_c = 0,$$

and therefore the conditions of the system equilibrium look as follows

$$\delta z_c = 0.$$

Thus, the positions, where the center of gravity occupies the lowest, highest or any other stationary vertical position (Torricelli principle) will be the positions of equilibrium of the system of heavy bodies.

One alternative proof of the principle of possible displacements (velocities) was presented by Lazare Carnot (1753-1823). The first scientific work of Carnot was treatise The Experience in Machines in
General, published anonymously in 1783. The third edition of the treatise was extended and renamed as Basic Principles of Balance and Motion. The main content of this work was to obtain, under the machine equilibrium, by calculating the force work increment in virtual displacements of the force application points (the term "work" was introduced later in the 19th century.). Carnot comes to the system of loads or weights; thus obtained system equilibrium is treated on the basis of Torricelli principle of the lowest position of its center of gravity. Like Torricelli, Carnot records the equation of extremity of vertical coordinate of the center of gravity instead of the condition of minimal height of center of gravity of the system of loads:

$$\int F \cdot u \cos z = 0,$$

where \(\int\) – the symbol of sum. This equality is perhaps one of the first analytical formulations of the principle of virtual velocities.

The idea of introduction of a replacing system in the 18th century proved fruitful, it was used in analytical mechanics by numerous contemporaries of L. Carnot – J.-L. Lagrange, J. Fourier, A. Ampere and others. The replacing systems (blocks, levers, hoists, etc.), when deriving a principle of possible displacement, in particular, evidence for a close relationship of this principle with real machines and mechanisms.

L. Carnot introduced the concept of "geometric motion", i.e. one that is allowed by constraints (ideal, unilateral). In modern terminology virtual displacements of the system points correspond to geometric motions.

In 1594, Galileo Galilei had written the famous treatise on mechanics «Delia scienza meccanica» (On Science Mechanics). Various statics problems were solved in it by using the principle of virtual displacements.

Galileo’s findings on proportionality in the organic world and technology, and ways to creation of a new Newtonian dynamics outlined by the scientist are of interest.
J.-L. Lagrange in his book *Analytical Mechanics* wrote: "The dynamics is the science of slowing down and accelerating forces and variable motions which they have to cause. This science is quite owes its development to new scientists, and Galileo is the person who has laid its first foundations. Before Gallilei they considered forces acting on the body only in equilibrium, and although the accelerated fall of solids and curvilinear motion of thrown bodies could not be attributed to any other reason, than the constant action of gravity, but nobody before Galilee was unable to determine the laws of these everyday phenomena, though their cause was so simple. Galileo was the first to make this important step and thus opened a new and boundless way for the progress of mechanics. His discovery was expounded and developed in the work «Discorsi e dimonstrazioni matematiche intorno a due nuove scienze», which appeared first in Leiden in 1638. However, this work had not brought Galilei so much fame among contemporaries as the discoveries made by him in astronomy; now it presents the most reliable and essential part of the glory of this great man."

In 1586 there appeared a treatise by S. Stevin *Principles of Statics*, where he states the principle of the impossibility of perpetual motion, gives the original proof of body equilibrium conditions on an inclined plane, opens the law of composition of forces (parallelogram of forces) and force resolution into two components, perpendicular to one another, formulates the principle of possible displacements for a particular case. The ancient statics was completed in this paper.
Finally, the principle of possible displacements (velocities) took its central place in mechanics on the appearance of the treatise *Analytical Mechanics* by Lagrange. Its first edition was published in 1788 in Paris [Lagrange, 1788] and consisted of two parts: *Statics* and *Dynamics*. Lagrangian general formula of dynamics (combination of D'Alembert principle and the principle of possible velocities) embraced all possible cases, including the mechanics of a material point, of absolutely rigid body and the mechanics of the system of material points with constraints, fluid mechanics, the problem of motion and equilibrium of elastic bodies etc. Methods of Lagrangian mechanics greatly influenced the subsequent development of science.

"*Static is the science of body equilibrium*", – wrote Lagrange at the treatise beginning and presents the historical analysis of this science. Having examined the different approaches to the lever principle (Archimedes, Galillei, Stevin, Huygens), he prefers the approach of Archimedes. The principle of composition and resolution of forces was first clearly formed Stevin. Lagrange found a close relationship between the principle of motion, clearly formulated by Galillei, and the principle of composition of forces following the rule of parallelogram. Proof of the parallelogram of forces based on previously known principle of composition of velocities or motions were first given by Newton and Varignon (P. Varignon) independently of one another in 1687.

Lagrange attaches much importance to XVI Varignon lemma («Nouvelle mecanique»), which is now called the Varignon theorem and is one of the main theorems of the slip vectors. According to this theorem, if the system of slip vectors \( \mathbf{F}_v \) is reduced to a resultant \( \mathbf{F} \), the resultant moment relative to a point 0 (or axis \( Z \)) is the sum of the system vector moments relative to the same point (or axis)

\[
\text{mom}_0 \mathbf{F} = \sum_v \text{mom}_0 \mathbf{F}_v \quad \text{mom}_1 \mathbf{F} = \sum_v \text{mom}_1 \mathbf{F}_v .
\]

The theorem was established by P. Varignon in 1687 for the case of convergent force system.
We should be grateful to Roberval and Varignon who summed up and compared three fundamental concepts of statics: lever principle, the principle of virtual velocities and the parallelogram of forces. In his book «Nouvelle mecanique» (published in 1687 and completed posthumously in 1725) Varignon – to some extent still in the Aristotelian tradition – uses the idea of equivalence of the lever, the principle of virtual velocities and that of the parallelogram of forces.

Lagrange formulated the principle of virtual velocities: "If any system of any number of bodies, or pixels, each of them affected by any forces, is in equilibrium and if this system is set in any small motion, following which each point will pass the infinitesimal way, representing its virtual velocity, the sum of forces, each multiplied, respectively, by the path covered by the point in the direction of the force to which it is applied, will always be zero, if small paths towards the force are considered positive, and the paths in the direction opposite to the force direction are considered negative”[Lagrange, 1788].

Lagrange considers this principle as the most general one in statics, that, in his opinion, was first realized by I. Bernoulli. The principle of virtual velocities is associated with Torricelli’s principle, which is that "if two loads are related to each other and are in such a state that their center of gravity cannot drop below, they are in equilibrium in this state" [Lagrange, 1788]. The principle of virtual velocities, according to Lagrange, "gave rise to the emergence of another principle" proposed by Maupertuis in 1746 under the title of "law of rest", which was later developed and generalized by L. Euler.

Lagrange believed that most important in this principle is that it "is not only very simple and general, but in addition, has the valuable and inherent in it advantage over other principles that can be expressed by the general formula which embraces all the problems that may arise concerning the body equilibrium”. Lagrange presents a proof for this principle by introducing a replacing system of polyspasts.
You should specifically focus on the theorem by J.-L. Lagrange, which he called "property of equilibrium, which refers to the maximum and minimum" (Section III of Analytical Mechanics). Lagrange considers that the equilibrium of a conservative mechanical system with a finite number of degrees of freedom, all the forces acting on the system, is potential, i.e. there is a function of potential energy of the system such that

$$Q_i = -\frac{\partial \Pi}{\partial q_i}, \quad (i=1,\ldots,n),$$

(5.5)

where $Q_i$ – generalized forces, $q_i$ – generalized coordinated, $n$ – a number of degrees of freedom of the system.

Since in the state of equilibrium all generalized forces $Q_i$ must be zero, by the virtue of justice (5.5) in this state is zero and the differential of potential energy $d\Pi = \sum_{i=1}^{n} \frac{\partial \Pi}{\partial q_i} dq_i$, and this, in turn, means that under the system equilibrium its potential energy has a stationary value

Lagrange in §5, Vol. 1 of Analytical Mechanics (1788) examined in detail the case, when the potential energy is not just of a stationary value, but of strictly minimum value. For this purpose, he developed the function $\Pi$ at equilibrium as a power series in small quantities $x_i$ that are increments of system coordinates
\[ \Pi = A + B_1 x_1 + B_2 x_2 + \ldots + C_1 x_1 x_1 + C_{12} x_1 x_2 + C_{22} x_2 x_2 + \ldots \] 

(5.6)

and offered to be limited to only the second degrees of variables \( x_i \).

It follows from zero equality of the value \( a_{ij} \) at zero values of \( x_i \), that \( B_j = 0 \). It is not difficult now to put the expression (5.6), using linear transformation, into the form

\[ \Pi = A + D_1 y_1^2 + D_2 y_2^2 + \ldots \] 

(5.7)

Then we use the fact that, if \( x_i = 0 \), then \( y_i = 0 \), and that means that in the state of equilibrium the function \( \Pi \) will have a local minimum \( \Pi_{\min} = A \) only if the coefficient \( D_i \) is positive.

Now let the system be recovered from the equilibrium state by attaching its masses \( m_1, m_2, \ldots \) initial small velocities \( V_1, V_2, \ldots \), i.e. the system is given certain kinetic energy \( T_0 = \frac{1}{2} (m_1 V_1^2 + m_2 V_2^2 + \ldots) \).

After recording under the law of conservation of total mechanical energy of the conservative system that Lagrange calls the principle of conservation of living forces

\[ \frac{1}{2} (m_1 V_1^2 + m_2 V_2^2 + \ldots) + A = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2 + \ldots) + A + D_1 y_1^2 + D_2 y_2^2 + \ldots, \] 

(5.8)

where \( v_1, v_2, \ldots \) -- mass velocity of the system in deviated state, It is seen that under such initial perturbation the velocities and displacements of the system points will be constrained by conditions

\[ (m_1 v_1^2 + m_2 v_2^2 + \ldots) \leq (m_1 V_1^2 + m_2 V_2^2 + \ldots); \quad D_1 y_1^2 + D_2 y_2^2 + \ldots \leq \frac{1}{2} (m_1 V_1^2 + m_2 V_2^2 + \ldots). \] 

(5.9)

According to Lagrange "It follows that in this case the system will be able to only slightly deviate from its equilibrium position and will be able to perform only very small oscillations with limited amplitude." In other words (Lagrange's theorem), if in some position of the system the potential energy has a strict minimum, that is the position of stable equilibrium.

In 1846, German mathematician Peter Gustav Lejeune Dirichlet published a paper On the Stability of Equilibrium (Über die Stabilität des Gleichgewichtes); he indicated that the proof of the stability of
equilibrium proposed by Lagrange and based on the representation of the function of potential energy by the initial segment of power series ". . . is not strict. Indeed, it is reasonable to doubt that the values for which we have small limits proceeding from the assumption that these values will always be very small (because we do it only when we can neglect the terms of the highest order), really always will remain for any period of time at these limits and, moreover, generally in small limits." Further in this paper Dirichlet has given a strict proof of the Lagrange theorem on stability of equilibrium without using the assumption of the possibility of the series expansion of the function $\Pi$ in terms of degrees of coordinates. This proof is based on the assumption which can be made without detriment to generality and which lies in the fact that all generalized coordinates and the potential energy of the system are zero in the state of equilibrium.

Dirichlet’s considerations are as follows. If the initial perturbed state is set by coordinates $q_{01}, q_{02}, \ldots$ and velocities $V_1, V_2, \ldots$, the law of conservation of energy can be written as

$$\frac{1}{2}(m_1 V_1^2 + m_2 V_2^2 + \ldots) = \frac{1}{2}(m_1 V_1^2 + m_2 V_2^2 + \ldots) + \Pi(q_{01}, q_{02}, \ldots) - \Pi(q_1, q_2, \ldots). \quad (5.10)$$

Since in terms of the assumption at $q_i = 0$ the potential energy $\Pi(q_1, q_2, \ldots)$ is zero and the minimum, you can choose sufficiently small positive values $l_1, l_2, \ldots$ such that $\Pi(q_1, q_2, \ldots)$ will have a positive value, if $|q_i| \leq l_i$, and if all $q_i$ are not equal to zero simultaneously. We consider the following variants of $q_1, q_2, \ldots$ when at least one of the coordinates $q_i$ is equal to its limit $l_i$. Of all values of the function $\Pi(q_1, q_2, \ldots)$ calculated at the considered variants of coordinates the smallest $p$ is chosen; then it is easily proved by contradiction that if initial displacements are limited by values $l_1, l_2, \ldots$, and if the initial perturbation is subject to inequality

$$\frac{1}{2}(m_1 V_1^2 + m_2 V_2^2 + \ldots) + \Pi(q_{01}, q_{02}, \ldots) < p,$$
then each of the generalized coordinates $q_i$ cannot exceed the absolute value of boundaries $i$, throughout all time of motion.

Hence the velocity $v_i$ should be satisfied by inequality

$$\frac{1}{2}(m_1v_1^2 + m_2v_2^2 + ...) \leq \frac{1}{2}(m_1v_1^2 + m_2v_2^2 + ...) + \Pi(q_{01}, q_{02}, ...) < p.$$  

The final conclusion made by Dirichlet, is "that limit of each velocity, as well as the limits of each variable $q_1, q_2, \ldots$ can also be arbitrarily small, since the values $l_1, l_2, \ldots$ can be arbitrarily small", and this means stability of the given state of equilibrium.

![René Descartes, Renatus Cartesius (1596 - 1650)](image)
![Joseph Louis Lagrange (1736 - 1813)](image)
![Johann Peter Gustav Lejeune Dirichlet (1805 - 1859)](image)

Note summing up that even in the present Dirichlet’s considerations with slight variations appear in the textbooks of analytical mechanics to prove the Lagrange theorem.

Thus, the principle known as Lagrange-Dirichlet one has been formulated: stable, unstable and indifferent equilibrium takes place, respectively, in the following situations.

<table>
<thead>
<tr>
<th>Stable equilibrium</th>
<th>Unstable equilibrium</th>
<th>Indifferent equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \Pi = 0$</td>
<td>$\delta \Pi = 0$</td>
<td>$\delta \Pi = 0$</td>
</tr>
<tr>
<td>$\delta^2 \Pi &gt; 0$</td>
<td>$\delta^2 \Pi &lt; 0$</td>
<td>$\delta^2 \Pi = 0$</td>
</tr>
<tr>
<td>min</td>
<td>max</td>
<td>const</td>
</tr>
</tbody>
</table>

263
If the system is in a potential field and, therefore, forces $\mathbf{F}_i$, acting on the system points, have the force function $W = W(x_i, y_i, z_i)$, then the sum of elementary works of force $\mathbf{F}_i$ on any displacement of the system will be full differential of the function $W$, which depends on $3n$ coordinate points of the system. In this case

$$dW = \sum_{i=1}^{n} (X_i dx_i + Y_i dy_i + Z_i dz_i),$$

where $X_i$, $Y_i$, $Z_i$ – projections of forces $\mathbf{F}_i$ on the axis coordinate.

It follows that

$$X_i = \frac{\partial W}{\partial x_i}, \quad Y_i = \frac{\partial W}{\partial y_i}, \quad Z_i = \frac{\partial W}{\partial z_i}$$

and therefore, the expression for the generalized forces $Q_j$ will look like:

$$Q_j = \frac{\partial W}{\partial q_j}.$$

Potential energy $U(x_i, y_i, z_i)$ of the system is defined as work that is to be done by the field forces to transfer the system from the positions $(x_i, y_i, z_i)$ to the zero position $(x_{i0}, y_{i0}, z_{i0})$, which, generally speaking, can be chosen arbitrarily. So

$$U(x_i, y_i, z_i) = \sum_{i=1}^{n} \int_{(x_i, y_i, z_i)}^{(x_{i0}, y_{i0}, z_{i0})} (X_i dx_i + Y_i dy_i + Z_i dz_i).$$

Changing the order of adding and integrating and using force function $W(x_i, y_i, z_i)$, we get

$$U(x_i, y_i, z_i) = \int_{(x_i, y_i, z_i)}^{(x_{i0}, y_{i0}, z_{i0})} dW.$$

After integration we will have

$$U(x_i, y_i, z_i) = W(x_{i0}, y_{i0}, z_{i0}) - W(x_i, y_i, z_i),$$

i.e. the potential energy $U$ with an accuracy to additive constant is equal to power function $W$, taken with the opposite sign

$$U(x_i, y_i, z_i) = - W(x_i, y_i, z_i)$$

(5.11)

264
Write now the principle of virtual displacements using potential energy $U$. We have
\[ \sum_{i=1}^{n} (X_i \delta x_i + Y_i \delta y_i + Z_i \delta z_i) = \sum_{i=1}^{n} \left( \frac{\partial W}{\partial x_i} \delta x_i + \frac{\partial W}{\partial y_i} \delta y_i + \frac{\partial W}{\partial z_i} \delta z_i \right) = \delta W = 0, \]
Or on the basis of (5.11)
\[ \delta U = 0, \tag{5.12} \]
i.e. the first variation of the potential energy $U$ must be zero, and this is the condition of its stationarity. Thus, the necessary and sufficient condition for the system equilibrium coincides with the condition of stationarity of the function $U$.

The principle of virtual displacement is a variational principle, because we do not consider here one system configuration but a set of possible configurations obtained as a result of virtual displacements admitted by constraints imposed on the system points.

The great advantage of this principle is that a set of all equilibrium conditions can be expressed by means of one equation, without going into the details of the constraints imposed on the system points. Formulations of the principle of virtual displacements do not include the reactions of constraints that eliminates the necessity to determine their values.

On the other hand, the reaction of constraints may be easily found using the principle of virtual displacements. To do this, we use the principle of eliminability. Rejecting the constraint, we replace its action by reaction, under these conditions, as was already noted, the number of degrees of freedom of the system increases. Considering then the system with lifted constraints, we give it a virtual displacement. Using further the principle of virtual displacements and equating the sum of virtual works to zero, including work of constraints reactions, we obtain one equation, which can help find the desired reaction of constraint.

The principle of virtual displacements allows obtaining all the conditions of equilibrium. It should be emphasized that the quantity of equilibrium equations obtained for the system may be equal to the number of virtual displacement realized in the system. In other
words, the number of equilibrium conditions that can be constituted for the system coincides with the number of its degrees of freedom.

A strict proof of the principle of possible displacements and its extension to the unilateral constraints was given by J. Fourier [Fourier, 1798], N.V. Ostrogradsky [Ostrogradsky, 1946].

J.B.J. Fourier has shown the ordinary expression of the principle of possible displacements in terms of potential energy (work of forces)

\[ \delta U = 0 \quad (\delta W = 0) \]  

valid for the so-called reversible displacements, that is the constraints in direction of these displacements can change a sign. In the case of irreversible displacements the equality (5.13) should be replaced by inequality

\[ \delta U \geq 0 \quad (\delta W \leq 0) , \]

and the usual formulation of the principle of virtual displacements – "the sum of all virtual works is zero" substitute "is zero" by "less than or equal to zero."

Jean-Baptiste Joseph Fourier was a French physicist and mathematician. Fourier’s only work in mechanics was *Memoir of Statics Containing a Proof of the Principle of Virtual Velocities and the Theory of Moments*. One of the provisions of this work was the consideration of cases of equilibrium forces applied to points of a mechanical system with so-called unilateral constraints (J.B.J. Fourier does not present such a term). J.B.J. Fourier considered as an example the equilibrium of two solids, their surfaces being pressed at the point of contact by two equal opposing forces, normal to both surfaces at the point of contact, the equilibrium of a flexible inextensible thread under the effect of two forces applied to its ends. J.B.J. Fourier stated (without a proof) that a necessary condition for thread equilibrium under the effect of such forces is nonnegativity of the "total moment of forces" on the virtual displacements of the points of their application. According to the terminology of that time the "total moment of forces" was called a sum of elementary works.
of all active forces on virtual displacement of the points of their
application taken with a negative sign. Thus, the condition of
equilibrium of the system of forces at unilateral constraints were
recorded in a form of a requirement of nonpositivity of the sum of
elementary works of all forces on virtual displacements. M.V.
Ostrogradsky in the development of the general theory of the
principle of possible displacements (1834) proceeded from the record
of this principle in memoirs of J.B.J. Fourier.

Note that the Fourier inequality $\delta U \geq 0$ corresponds to Young’s
inequality

$$f(x_1, x_2, \ldots, x_n) + H(p_1, p_2, \ldots, p_n) \geq \sum_{i=1}^{n} p_i x_i. \quad (5.15)$$

In the case of potentials of structural mechanics $U(\Delta)$ is the
potential energy of elastic deformation, $U^{\text{compr}}(P)$ – complementary
potential energy, the Young inequality looks as follows

$$U(\Delta) + U^{\text{compr}}(P) \geq \sum_{i=1}^{n} P_i \Delta_i.$$ 

The first variation on both sides of the above expression gives

$$\delta \Pi^L(\Delta) \geq 0, \quad | \delta \Pi^C(P) \leq 0,$$

$$\Pi^L(\Delta) = U(\Delta) - \sum_{i=1}^{n} P_i \Delta_i; \quad \Pi^C(P) = -U^{\text{compr}}(P) + \sum_{i=1}^{n} P_i \Delta_i.$$ 

Let the possible displacements be taken such that the system
maintains contact with all its bilateral supports and is separated from
one or more unilateral supports. Since the implementation of the
latter is always directed towards a possible displacement, their virtual
work in these displacements is always positive. Let it be $A_R$, then
$A_R \geq 0$. Virtual work of all other external forces denote as $A_P$. Then
the equilibrium condition of the system (the principle of possible
displacements) provides

$$A_R + A_P = 0.$$ 

Hence

267
\[ A_p = -A_R \leq 0. \]

Thus in the case of the system equilibrium, total virtual work of external forces is negative or equal to zero on those possible displacements, on which the loaded system is separated from one or more of its unilateral supports (in which unilateral constraints are excluded from work).

\[ A_p \leq 0. \]

With regard to the internal forces, which is identified with potential energy of the system, we have the expression of the principle of possible displacements

\[ U + A_p = 0, \]

but in the case of unilateral constraints, as was proved above \( A_p \leq 0 \). Then \( U \geq 0 \). If you pass to variations of works, i.e. works on infinitesimal displacements \( \delta w \), we get

\[ \delta U \geq 0 \]

and, respectively

\[ \delta A_p \leq 0. \]

Strictly speaking, all of the presented results are outstanding both in terms of content and time. But chronology of analogies may be represented as follows: G. Leibniz, who has a record of Euler’s theorem and Legendre transform; J.-L. Lagrange Analytical Mechanics (1778); Euler's theorem on homogeneous functions (1779); published Legendre transform (1787); Lagrange-Castigliano theorem and Fourier inequality, M.V. Ostrogradsky; Clapeyron theorem (published by Lame in 1852), and finally, generalized notion of duality according to Young, Young inequality; Young-Fennel inequality (Werner Fennel (1905 1988)). The question of the theory of systems with unilateral constraints expounded in the book by I.M. Rabinovich [Rabinovich, 1975].
It should be noted that J.-L. Lagrange in his book *Analytical Mechanics* (1788) pointed out that “The main feature of equilibrium, which is that any system of forces in equilibrium, continues to remain in this state, when each of these forces changes the direction of its action for the opposite, unless the structure of this system undergoes any change as a result of changes in the direction of all forces”. The last note proves that J.-L. Lagrange anticipated availability of such constraints and necessity of their consideration.

P. Appel has given a definition of this concept: "if possible displacements, compatible with the constraints, are set by irregularities, then constraints are called unilateral". We consider the general case where the constraints between the points are expressed by $h$ dependencies, including $g$ equalities and $h-g$ inequalities. Those of possible displacements of the system, where left parts not only of all the equations and inequalities are zero, Mr. Appel called displacements of equalities, others – equalities of irregularities. The following theorem has been proved. To balance the system, which is in the state where all constraints are included, it is necessary and sufficient that for all displacements compatible with the constraints, the sum of work of acting forces is equal to zero, or was negative; it was zero for displacements of equalities, negative for displacements of inequalities. The system even when all constraints are included is geometrically variable. Further, we assume that those constraints, which inequalities correspond to, are called unilateral.
Paul Emile Appell (1855-1930) was a French mathematician and engineer. He has derived ordinary differential equations describing motion of holonomous and nonholonomous systems (most general equations of motion of mechanical systems) that are called Appel’s equations. His Treatise of Rational Mechanics was published in 1833-1896.

5.3. First variational principles. Kepler, Fermat, Principia by Newton. Leibniz, Maupertuis

No general methods for solving extreme problems were developed before the 17th century and each problem was solved by specially developed method. In 1615 Johannes Kepler (1571-1630) published his book New Wine Barrels Stereometry [Kepler, 1935]. The author begins the book: "In November last year I brought to my home a new wife (first wife of J. Kepler, who gave birth their three children, died in 1610; in 1613 he married again and had eight children from his second wife); when Austria, that had comleted vintage distributed their products, sending loaded barges up the Danube, both our Noryk and the whole coast in Linz were blocked up with wine barrels that were sold at a reasonable price. Acting as a husband and a good father of the family, I had to take care of the necessary drinks. Therefore several barrels were brought and put in the house, and four days later a trader came with a measuring ruler and measured barrels in succession with no regard for their form, without any considerations and calculations. The copper ruler end moved through the pouring hole to the barrels wooden circle that we call at home the bottom, and after the length from the top to the belly and from the belly to the bottom was equal, he announced the number of amphorae to fill a barrel, noting a number on the line, at the point where the ending length is designated, and by this number he defined price."

J. Kepler was very much surprised, it seemed strange, how one can calculate having only one dimension the capacity of barrels of different shapes. "I thought it reasonable, he says, to take a new
object for mathematical study and examine geometrical laws of such convenient measurement and to determine its foundations."

In the course of solving the set problem J. Kepler laid the foundations of differential and integral calculus and formulated first general rules for solving extreme problems. It is believed that even the integral sign comes from the sums designated by J. Kepler. He writes: "Under the influence of charity genius, being undoubtedly a good geometer, coopers began to provide some form of barrels, which at a given length of the line measured by a measurer permits us to judge of the largest capacity of a barrel, and, since in the vicinity of each maximum changes are negligible, so small deviations do not have a significant effect on the vessel capacity." And further: "The figures on either side of their highest capacity show insignificant reduction" (Theorem V, annex).

There is a sufficiently general form of expression of the extremity criterion, which was later transformed into the exact theorem (first for polynomials) by Fermat (1629), and then – by Newton and Leibniz and called Fermat's theorem.

J. Kepler, in fact, does not use this criterion for finding the extreme dimensions and shapes of bodies, he is guided by other considerations. But he clearly understands the importance of generalizing properties, which became a source of differential calculus. It should be added that he is also the author of the interpretation of the problem on finding areas and volumes, which initiated the integral calculus. This shows exactly the role of J. Kepler in creating new mathematics.

Einstein, who called J. Kepler "incomparable man", wrote about his fate as follows [Einstein, 1965-1967, p. 121]: "He lived in an epoch when there was no confidence in the existence of some general laws for all natural phenomena. How deep was his belief in such a pattern if working alone, with no support and understanding he, for many decades, derived his strength for heavy and laborious empirical study of planetary motion and mathematical laws of this motion. Today, when the act of research has been realized, nobody
can fully appreciate the amount of inventiveness, hard work and patience needed to discover these laws and to express them so precisely."

A simple stone slab was placed on J. Kepler’s grave and it is not even known whether a Latin epitaph, expressed by J. Kepler and translated by the author of preface to the book *New Wine Barrels Stereometry* M. Ya. Vygodsky, was engraved on this slab:

*I measured heavens, now measure shadows of the Earth

My spirit now in the skies, and here the body shadow lies

If J. Kepler in his epitaph says that in his life-time his spirit existed in heaven, we can say that after his death, his spirit remains on the earth.

Perhaps the first clear formulation of the variational principle concerning physical problems was given in 1662 by the French mathematician Pierre de Fermat (1601-1665). That was the principle of the shortest time or "Fermat's principle."

It is known that the light refraction law was installed by Villebrord Snell (published under the Latinized name Snellius) and R. Descartes. In so doing Descartes made a number of assumptions, including the least justified one that the light velocity in a dense medium is higher than in a less dense medium. Then English philosopher Hobbes argued against this, and Fermat did it in 1662.

Fermat assumed the principle of the shortest time as a basis of studying the light refraction law. In the paper, "Synthesis ad Refractiones” (*Refraction Synthesis*) [Fermat, 1891, p. 173-179], he derived the law of light refraction using geometry, proceeding from this principle. According to Fermat’s opinion "Nature has the easiest and most attainable ways, and not shorter," as many people think. Specifying the idea, he said: "Like Galileo, who, considering the motion of heavy bodies in nature, measured the ratio of this motion rather by time than by distance, we consider not only the shortest distances or lines, but those that may be passed in more easy and convenient way and for a shorter interval of time."
As is known, Fermat's principle is the most general mathematical form of the laws of geometrical optics.

In fact Fermat has shown that Snellius law of refraction satisfying the hypothesis that the time taken for the trajectory of neighboring the valid one differs from the transit time of the latter by the value of the second order of smallness. The proof of Fermat really includes the statement that the variation (we mean here the variation, though the general concept of functional variation was introduced almost a hundred years later by J. L. Lagrange) of a certain defined integral, taken along some trajectory of the beam, is zero. This condition is necessary but insufficient for the time be minimal. In a simple case, considered by Fermat, the minimum condition and variation condition coincide, but in more complex ones this is not the case.

Fermat's principle has led not only to experimentally studied fact, but also to the new result that the refractive index is equal to the ratio of light velocities in two media. Fermat wanted to prove that his viewpoint that the light spreads more slowly in a more dense medium is true, while Descartes defended the opposite point of view. In any case, the principle of the least time was derived a priori, but not by induction.

The first real justification of Fermat principle was given by H. Huygens [Huygens, 1935], who, based on his "wave theory", proved that the refractive index at the boundary of two media is equal to the ratio of light velocities in these media. The H. Huygens proof shows that the time the light takes to cover the distance between two points is really minimal.

By the way, it is interesting that H. Huygens in 1657 patented the first pendulum clock.

Thus the shortest time principle was formulated in geometrical optics. Immediately and naturally there appeared a problem of searching for similar problems on a minimum value of time in mechanics. The object of this kind was the task given by Isaac Newton (1643-1727) in his «Philosophiae Naturalis Principia
Mathematica» (Mathematical Principles of Natural Philosophy, abbreviated as Principia) [Newton, 1936, p. 426-427], he presented its solution, without indicating the method used for its finding: what shape should be given a solid body of rotation moving along the axis for the resistance, which it perceives, be minimal. Principia were published in 1687.

Although that Newton’s approach inevitably resulted in the mechanics which we know today, Euler needed a larger part of life to understand and develop the concepts of Newton, to supplement them with no less important new ideas and demonstrate how one can solve real problems. L. Euler was a leading theoretical physicist of the 18th century. Although in the ordinary insignificant historical books and references his works were underestimated, the brief factual history of the old Handbuch der Physik includes twice more special discoveries of L. Euler than of any other physicist, who lived before him and after him.

In 1788, just one hundred years after the Principia there appeared Analytical Mechanics by J.-L. Lagrange which is a bit less famous. In general it was ardently described in any popular study of the history of science, where V.R. Hamilton’s words are quoted that it is "a kind of scientific poem."

The first general analytical method of solving extreme problems was developed by Fermat. It was probably developed in 1629, but was first sufficiently described in a letter to J. Roberval in 1638. To understand the primary Fermat thought one should turn to the book by Descartes, containing this letter [Descartes, 1938, p. 154]. In the modern language (although Fermat proves it only for polynomials) the Fermat method is reduced to the thought that, when finding
extremum of the function $f(x)$ without restrictions at the point of extremum $\bar{x}$, the equality $f'(\bar{x}) = 0$ should be fulfilled. It is known that the first hint of this result appeared in the words of J. Kepler in *New Wine Barrels Stereometry*.

Fermat’s considerations acquired their exact meaning 46 years later, when in 1684 there appeared the work by Gottfried Wilhelm Leibniz, which laid the foundations of mathematical analysis. The very title of this work, which begins: «Nova methodus pro maximis et minimis ...» (*The New Method of Finding the Largest and Smallest Values ...*), shows the importance of the problem of extremes in the development of modern mathematics. In his article G. Leibniz not only obtains, as a necessary condition, the ratio $f'(\bar{x}) = 0$ (this result is called Fermat's theorem), but uses the second differential for distinguishing between maximum and minimum. It should be noted that Newton already knew most of these Leibniz facts. However, his work *Method of Fluxions* was mainly completed before 1671, but it was published only in 1736.

Omitting the questions of priority, it should be noted that for the optical problem the value in geometrical optics, which must reach a minimum in specific phenomena is highly accessible and does not require further research. This is – time. It is not clear in mechanics that the magnitude in the process of motion should have either minimum or maximum. In the opinion of scientists, engineers of the 17th century nature always acts in the simplest way. The first rule of reasoning in Newton’s physics: "*Do not take other reasons in nature over those that are true and sufficient to explain the phenomena. In this regard philosophers state that nature does nothing in vain, and more in vain when less will serve, for nature is pleased with simplicity and affects not the pomp of superfluous causes*"[Newton, 1936]. On this occasion comes to mind expression of Antoine de Saint-Exupery: "*I seems perfection is attained not when there is nothing more to add, but when there is nothing more to remove.*"

Leibniz has formulated interesting hypothesis "*everything possible is committed to existence.*" With the collision of...
opportunities "a number of things, which contains the largest number of possibilities" is realized. This number is unique and defined as a straight line among lines, a right angle among angles, the most spacious figure among figures, namely a circle or ball. G. Leibniz postulated the existence of the principle of the greatest number of existence, that explains why if you have to go from one point to another, when the line direction is not defined, you choose the easiest and shortest way: if you should move from the possibility to reality, the existence quantity should be "as great as possible for a given possible order of existence" [Leibniz, 1982].

The problem of mechanics was to clarify which value may be minimum (or maximum) in the process of motion. This problem, like Fermat's principle, arose in the 17th century and was more or less clearly elucidated only in the middle of the 18th century. It was brought to the same mathematical clarity and certainty as Fermat's principle, only at the end of the 18th-early 19th century. The concept of action was first formulated by Leibniz, and in this respect P.-L. Maupertuis refers to him.

L. Euler believed that "in the world there is nothing, where noticeable sense of a maximum or minimum would be observed."

Already in the 20th century German mathematician Carl L. Siegel allowed himself a saying: "According to Leibniz our world is the best of all possible worlds, and therefore the laws of nature can be described by variational principles."
The variational principle in mechanics was first formulated in 1744 by Pierre-Louis Moreau de Maupertuis (1698-1759).

On April 15, 1744 (i.e., a few months before the appearance of the book by L. Euler Method of Finding Curves with the Properties of Maximum or Minimum, or Solution of Isoperimetric Problem in the Broadest Sense), a former French officer P.-L. Maupertuis presented the Paris Academy his memoir «Accord de differentes lois de la Nature qui avaient jusqu'ici pari incompatibles» (Reconciliation of Various Laws of Nature That Were Yet Considered Incompatible) [Maupertuis, 1744, p. 571, 1756, p. 3-28]. Maupertuis says primarily about the propagation of light. Even earlier, in 1740 P.-L. Maupertuis said that in the simplest cases of equilibrium, a certain function of forces has its maximum or minimum [Maupertuis, 1740, p. 240]. This law was then reviewed in 1748/49 by Courtivron (1715-1785) [Courtivron, 1749, p. 21 and more] and in 1751 by Euler.

Only in 1746 P.-L. Maupertuis proclaimed the universal law of motion and equilibrium – the principle of the least amount of action. The term amount of action he understood in the sense of activity and measured it as the product \( mv \), where \( m \) – mass, \( v \) – velocity, \( s \) – path, covered by the body.\(^{27}\)

According to P.-L. Maupertuis for motion \( mv = \text{min} \), and in the case of equilibrium the state of the body is such that, when it is given small motion, then the amount of action is minimum.

It may be considered that the most significant contribution of Maupertuis is interpretation of the principle of a minimum momentum as a universal law of nature, while L. Euler considered the same relationship as more meaningful and mathematically exactly expounded, and as that fit for only partial problems. That was universal understanding of the principle of the least action formulated P.-L. Maupertuis which caused Euler’s recognition of P.-L. Maupertuis’ priority. Neither Leibniz nor L. Euler had such universal

\(^{27}\) P.-L. Maupertuis, like Descartes, believed that the mechanics of the main parameter is momentum \( mv \).
principle, although the same principle, but not raised to the rank of "the world creation laws" was deduced by Euler even before P.-L. Maupertuis.


If we consider a mechanical system of solids, then the principle of possible displacements as well as the statics equilibrium equation allows finding the effects of the external force on the mechanical system. The number of equations, based on the principle of possible displacements, equals the number of degrees of freedom of the mechanical system.

In mechanics, the degrees of freedom are a set of independent coordinates of displacement and / or rotation that completely define the position of the system or the body (along with their time derivatives – appropriate velocities – completely determine the state of a mechanical system or body – that is, their position and motion). This fundamental concept is used in theoretical mechanics, structural mechanics, the theory of mechanisms and machines, mechanical engineering, aviation and theory of aircrafts, robotics and other fields.

It should be noted that several theorems, being of fundamental importance for statics of hinged-bar systems (trusses) have been formulated by A.F. Möbius, professor of astronomy at Leipzig University. He considers in his manual in statics [Möbius, 1837, Vol.2, hl.4,5] the problem of equilibrium of the system of hinged bars, and shows that, if the total number of hinges in such a system is \( n \), then to obtain a fixed stationary system of bars joining these hinges should have at least \( 2n-3 \) bars for a plane system and at least
3n-6 bars in the case of spatial system. Thereat A.F. Möbius points to the possibility of exceptional cases, when the system of 2n-3 bars cannot be absolutely rigid, admitting the possibility of small relative displacements of hinges. Examining these exceptional cases, he finds that the determinant of the system of equilibrium equations for such trusses becomes zero.

Unfortunately, the important work by A.F. Möbius remained unknown for many years, and only when the practice has mastered the use of steel trusses and when in this context there was a need to improve their general theory, engineers rediscovered the theorems by A.F. Möbius. The outstanding role in this rediscovery belongs to Otto Mohr [Mohr, 1874, p. 509; 1885, p. 289; 1905, ch. XII]. He established the number of bars required to form a rigid statically determinable system studied in this exceptional case of infinitesimal mobility. He has proved that there are statistically detectable trusses that cannot be calculated by previously developed methods and proposed to solve these systems by the method of possible displacements.

A somewhat different version of the method of possible displacements was proposed by Müller-Breslau [Müller-Breslau, 1887, Vol. 9, p. 121].

It should be noted that the principles of the general theory of space systems were laid by A.F. Möbius. In particular, he showed that to join n hinges in a rigid geometrically stationary system one should have 3n-6 bars, noting that there may be exceptional cases of infinitesimal mobility and they are characterized by zero determinant of the system of equilibrium equations for all nodes. He indicated a useful practical method for settling the question of whether the system is rigid or not – if for any load we can find forces in all elements of the system without coming to indeterminateness, then the above determinant is nonzero and the system is indeterminate. As the simplest assumption A.F. Möbius admits loading by zero forces, and provided that in none of the bars the force is not different from zero, the system is rigid.
A.F. Möbius investigated a very important problem of self-balanced spatial bar system [Möbius, 1837, v.2, p. 122] in the form of a closed polyhedron and showed that, if the plane faces of the polyhedron is a triangle or composed of triangles, the number of bars in it is exactly equal to the number of the equations of statics, and this system is statically indeterminable. Fig. 5.1 presents examples of such systems.

The works by A.F. Möbius in spatial systems also remained unknown to engineers, and they soon developed a theory of such type trusses independent of Möbius. This was mainly done by A. Föppl, who combined his studies on the subject in his published book [Föppl, 1892]. In this book we find first the development of some important questions concerning spatial systems. The book was a significant contribution made by Föppl and became the basis for many of following works in this field.

P.L.Chebyshev laid the foundations of the theory of structure of plane mechanisms. In the work On Parallelograms [Chebyshev, 1870] he has derived a structural formula for linkage mechanisms with turning kinematic pairs and one degree of freedom (now known as the Chebyshev formula [Chebyshev, 1948]) – an identity that should be satisfied by any such mechanism

$$3m - 2(n + \nu) = 1,$$

where $m$ – the number of moving parts, $n$ and $\nu$ – numbers of respectively movable and immovable joints. After 14 years, this formula was rediscovered by German engineer M. Grübler [Tyulyna, 1979]. In 1887 Chebyshev’s student P.O. Somov obtained a similar structural formula for spatial mechanisms [The History of Mechanics in Russia, 1987].

280
When the number of degrees of freedom of a kinematic chain in this case means the number of degrees of freedom of mobile links relatively to a base (a link taken as fixed). However, the base itself can move in the real space.

However, whether the machine is moving or not, the character of motion of the members of reciprocating engine relative to stationary base remains unchanged.

Introduce the following notations:

\( k \) – the number of links in the kinematic chain
\( p_1 \) – the number of kinematic pairs of the first class in this circle
\( p_2 \) – the number of pairs of the second class
\( p_3 \) – the number of pairs of the third class
\( p_4 \) – the number of pairs of the fourth class
\( p_5 \) – the number of pairs of the fifth class.

The total number of degrees of freedom \( k \) of free links arranged in space, is equal to \( 6k \). In the kinematic chain they are coupled in kinematic pairs (i.e. constraints are imposed on their relative motion).

In addition, a kinematic chain having a base (link, taken as fixed) is used as a mechanism. That is why the number of degrees of freedom of the kinematic chain will be equal to the total number of degrees of freedom of all links except for constraints imposed on their relative motion:

\[
W = 6k - \sum S_i
\]

The number of constraints imposed by all pairs of class I, is equal to their number, because each pair of the first class imposes a constraint on the relative motion of the links coupled into a pair; the number of constraints imposed by all pairs of the second class, is equal to the doubled number (each pair of the second class imposes two constraints), etc.

In the link taken as fixed all six degrees of freedom (six constraints are imposed on a base) are subtracted. So:

\[
S_1 = p_1, \quad S_2 = 2p_2, \quad S_3 = 3p_3, \quad S_4 = 4p_4, \quad S_5 = 5p_5, \quad S_{base} = 6
\]

281
and the sum of all constraints

$$
\sum S_i = p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 + 6.
$$

As a result we obtain the following formula to determine the number of degrees of freedom of a spatial kinematic chain:

$$
W = 6k - p_1 - 2p_2 - 3p_3 - 4p_4 - 5p_5 - 6.
$$

Grouping the first and last terms of the equation, we get:

$$
W = (6k - 1) - p_1 - 2p_2 - 3p_3 - 4p_4 - 5p_5,
$$

or finally:

$$
W = 6n - p_1 - 2p_2 - 3p_3 - 4p_4 - 5p_5,
$$

where \( n \) – the number of mobile links of kinematic chain

This equation is called structural formula of kinematic chain of the general form.

The formula was obtained for the first time (in a slightly different form) by P.O. Somov in 1887 and developed by A.P. Malyshev in 1923. Thus it is often referred to as the Somov-Malyshev formula. In some books it is called Malyshev formula – by final authorship.

The theory of mechanisms and machines began its formation as science in the late 18th-early 19th centuries entitled *Applied Mechanics*.

However, machines existed long before that date. Therefore, the history of the theory of machines and mechanisms (TMM) can be conventionally divided into four periods:

The 1st period prior to the beginning of the 19th century – the period of the empirical mechanical engineering during which the great number of simple machines and mechanisms were invented: elevators, mills, stone crushers, weaving and turning machines, steam machines (Leonardo da Vinci Waist, Polzunov Watt). At the same time the theory foundations are laid: theorem of change of kinetic energy and mechanical work, "the golden rule of mechanics" laws of friction, the concept of reduction, principles of geometric theory of
cycloid and involute gearing (Carnot, Coulomb, Amontons, Cardano, Roemer, Euler).

The 2nd period from the beginning to the middle of the 19th century – the period of early development of TMM. That is the time of development of such sections as kinematic geometry of mechanisms (Savary, Shawl, Olivier) kinetostatics (Kariolis) flywheel design (Poncelet) classification of mechanisms as to the motion transformation function (Monge, Lanu) and other sections. First scientific monographs on the machine mechanics have been written (Willis), first courses of TMM were read and first textbooks were published (Betancourt, Chizhov, Weisbach).

The 3rd period – from the second half of the 19th century to the beginning of the 20th century – the period of fundamental development of TMM. During this period scientists developed: fundamentals of structural theory (Chebyshev, Grübler, Somov, Malyshev), principles of the machine control theory (Vyshnegradsky), grounds of the theory of hydrodynamic lubrication (Grübler), basics of analytic theory of gearing (Olivier, Gokhman), principles of graph-analytical dynamics (Wittenbauer, Mertsalov), structural analysis and structural classification (Assur) method of plans of speeds and accelerations (Mohr, Manke), mechanism idling rule (Grashof) and many other sections of TMM.

Academician P.L. Chebyshev (1821-1894) – the famous Russian scientist, mathematician and engineer – published a series of papers on the structure and synthesis of linkage mechanisms. Using his method, he invented and designed more than 40 new mechanisms, performing specified motion trajectories, stop of links under other motions, etc. He is rightly considered the founder of the Russian school of the theory of mechanisms and machines, and the structural formula of plane linkage mechanisms is called Chebyshev formula.

Structural group of Assur (also a group of Assur) is the shortest kinematic chain formed by the lower pairs of the fifth class; when it is coupled with some plane mechanism, its degree of mobility does not change.

283
The group is named after L.V. Assur, who has developed the method of their formation in the early 20th century. [Assur, 1913-1914].

The Assur groups are divided into classes, types and orders.

- the Assur group class is defined by the class of the highest circuit that is included in it.
- the Assur group type is defined by a combination of rotational (hinges) and translational (sliders) of kinematic pairs in the group.
- the Assur group order is defined by the number of kinematic pairs by which the group is attached to the mechanism.

Structural group of $n = 2$ and $p = 3$ is called a two-lead group.

The 4th period from the beginning of the XX century to the present – a period of intense development of all trends of TMM

**5.5. Dynamics. Principles of D'Alembert, Jourdain, Gauss, Hertz**

Three approaches are used in dynamics when deriving motion equations. The first two are based on the differential principles of mechanics – d'Alembert principle and the principle of possible displacements; the third approach – on the Hamilton integral principle.
One of widely used approaches of constructing motion equations is the method based on the d'Alembert principle. In accordance with the d'Alembert principle, if the assigned active forces acting on the point of the mechanical system, and the reactions of imposed constraints are added inertia forces we obtain a balanced system of forces.

In 1743 Jean Le Ron D'Alembert (1717-1783) – French scholar and encyclopedist, commonly known as a philosopher, mathematician and engineer, published a *Treatise on Dynamics*, which was the first work, which formulated general principles of derivation of differential equations of motion of material systems, and dynamic problem were reduced to the problems of statics. J.L.R. D'Alembert has essentially extended the use of the principle of virtual displacements to dynamics. Basic mathematical research of J.L.R. D'Alembert refers to the theory of differential equations. His work, along with studies of L. Euler and D. Bernoulli became the basis of mathematical physics.

Note that the original idea of the principle formulated by J.L.R. D'Alembert was expressed by Jacob Bernoulli, when studying the problem of the center of oscillation of bodies of arbitrary shape. In 1716 the St. Petersburg Academician J. Herman advanced a principle of static equivalence principle of "free" motions and "actual" motions, i.e. motions performed in the presence of constraints. Later this principle was applied by Euler to the problem of vibrations of flexible bodies (this work was published in 1740) and was called *The Petersburg Principle*. But J.L.R. D'Alembert was the first to formulate the considered principle in a general form, though had not given it a certain analytical expression. The analytical expression of
this principle was given later by J.-L. Lagrange in his *Analytical Mechanics*. Interestingly, J.L.R. D'Alembert in the *Treatise on Dynamics* [D'Alembert, 1743], when expounded the principle, which was later called the *D'Alembert principle*, did not use the term *inertial forces*. The term *d'Alembert inertial forces* came much later. Euler thought that the term *inertial forces* was first introduced by J. Kepler, who considered them "body’s force of resistance to everything that tries to change its state of motion".

If the proposed mechanical system has a complex structure and consists of discrete masses and bodies of finite dimensions, it is very difficult to record the dynamic equilibrium equation. In this case, it is often efficient to use the principle of possible displacements (d'Alembert-Lagrange principle). The principle of possible displacements or general equation of dynamics is formulated in the problems of dynamics as follows: the system motion with ideal constraints occurs in such a way that at any time the sum of works of all active forces and inertial forces in any possible displacements is zero.

The principle of possible displacements is equivalent to the equation of dynamic equilibrium, but the variational formulation is much more extensively used in mechanical problems. The fact is that the force works on possible displacements are scalar values and can be added algebraically, whereas forces themselves are vectors and should be added following the rules of vector analysis.

Note that in analytical mechanics the D'Alembert-Lagrange principle is often written in the form

\[ \sum_{i=1}^{n} \left[ (X_i - m_i \ddot{x}_i) \delta x_i + (Y_i - m_i \ddot{y}_i) \delta y_i + (Z_i - m_i \ddot{z}_i) \delta z_i \right] = 0. \]

Also note that in analytical mechanics they sometimes also use other variational principles like the principle of possible displacements – the *Jourdain* (Philip Edward Bertrand Jourdain) principle and the *Gauss* (Johann Karl Friedrich Gauss) principle. Give in brief their formulation as variational equations.
The Jourdain principle

$$\sum_{i=1}^{n} \left[ (X_i - m_i \ddot{x}_i) \delta x_i + (Y_i - m_i \ddot{y}_i) \delta y_i + (Z_i - m_i \ddot{z}_i) \delta z_i \right] = 0.$$ 

The Gauss principle (principle of the least constraints)

$$\sum_{i=1}^{n} \left[ (X_i - m_i \ddot{x}_i) \delta x_i + (Y_i - m_i \ddot{y}_i) \delta y_i + (Z_i - m_i \ddot{z}_i) \delta z_i \right] = 0.$$ 

Using the Gauss principle we can get differential equations of motion of a mechanical system with ideal constraints, in particular it follows that in the absence of set forces the point will move along this plane surface on the curve that has the smallest curvature. This shows the connection of Gauss principle with the principle of straight-line way of Heinrich Rudolf Hertz.

Hertz principle\(^{28}\) (principle of the least curvature) – one of variational principles of mechanics, according to which the trajectory, which has the smallest curvature or "most straight one will be actual in the absence of active forces of all kinematically possiblenes, that is admissible constraints of trajectories”.

\[\text{Jean Le Rond D'Alembert, d'Alembert (1717–1783)}\]
\[\text{Philip Edward Bertrand Jourdain (1879-1919)}\]
\[\text{Johann Carl Friedrich Gauß (1777-1855)}\]
\[\text{Heinrich Rudolf Hertz (1857-1894)}\]

While giving credit to the Jourdain, Gauss and Hertz principles, however, note that the most universal and convenient in terms of theoretical and practical application is probably the d'Alembert-

\(^{28}\) See G. Goldstein: \textit{Classical Mechanics} [Goldstein, 1957].

287
Lagrange principle. Sam Johann Carl Friedrich Gauss, before displaying the principle of the least constraint, wrote [Gauss, 1829]: "As is well known the principle of virtual velocities transforms all statics into a mathematical assignment, and by D'Alembert's principle for dynamics, the latter is again reduced to statics... It follows that neither fundamental principle of equilibrium and motion can differ significantly from the two above principles, and that whatever this principle be, it can always be considered as a more or less direct conclusion from the above ones."

5.6. The principle of Hamilton-Ostrogradsky. The dual principle of Hamilton-Poincaré

The studies of Irish mathematician W. Hamilton have become a significant milestone in the history of variational principles prepared by the development of science and technology.

William Rowan Hamilton (1806-1865) – a prominent Irish mathematician and physicist of the 19th century. His main works are devoted to mathematical optics, mechanics, calculus of variations. He has established a general integral variational principle of classical mechanics for conservative systems (1833). This principle was generalized by M.V. Ostrogradsky (1850) for the non-conservative system (principle of Hamilton-Ostrogradsky). Quaternion calculus, which laid the foundations of the vector (and operational) calculus, and which is the first noncommutative algebra, Hamiltonian operator – these are major achievements that have entered Hamilton's name in the history of physics and mathematics.

For a long time W. Hamilton was interested in imaginary values, their geometrical interpretation and possible generalizations. In 1843 he came to the discovery of calculus of quaternions – the hypercomplex numbers. This is his main and most significant contribution to mathematics. On October 16, 1843, he established the fundamental theorem of multiplication of quaternions, underlying
noncommutative algebras. In November 1843 he made a report about the discovery at the Royal Irish Academy

That is how W. Hamilton wrote about it in a letter to his son on August 5, 1865: "... on the 16th day of the same month – which happened to be a Monday and a Council day of the Royal Irish Academy – I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an undercurrent of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, \( i, j, k \); namely,

\[
i^2 = j^2 = k^2 = ijk = -1\]

which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact, that I then asked for and obtained leave to read a Paper on Quaternions, at the First General Meeting of the session: which reading took place accordingly, on Monday the 13th of the November following".

In mechanics W. Hamilton is a direct successor of the trend of J.-L. Lagrange. This is reflected not only in his passion for Analytical Mechanics, which he called a "scientific poem", and not only that, he worked analytically without using visual geometric representations, even where they could provide him with direct assistance. The most important factor here is W. Hamilton’s opinion as to research problems in mechanics, which brings him to J.-L. Lagrange: mechanical problems are a class of mathematical problems,
development of mechanics is the development of mathematical methods.

Such a case is characteristic of W. Hamilton’s views. Someone once noticed: "I do not know people who have not seen a conical refraction, but would believe in its existence. I had attracted attention of two dozens of mathematicians, showing them the cone of light". W. Hamilton replied: "How is this different from my approach. If I only saw conical refraction, I would have never believed in it. My eyes often deceived me. I believe in conical refraction, because I have proved it" [Truesdell, 1980].

Hamilton's principle is an integral approach, by which one can implement effective algorithms for constructing equations of motion. This is done as variational principle both for finite-dimensional and continual dynamical systems. To simplify presentation, consider finite-dimensional dynamical systems [Bazhenov et al., 2012].

Let the state of some mechanical system is characterized by $n$ generalized coordinates $q_1(t), q_2(t), ..., q_n(t)$. Consider the $(n + 1)$-dimensional expanded coordinate space $q_1(t), q_2(t), ..., q_n(t)$ (Fig. 5.2). Let at time $t_1$ the mechanical system is at point $B$ that defines a certain dynamic status of the system. Suppose that the system undergoes some dynamic effects. As a result, the system state will evolve.

Generalized coordinates

$$\mathbf{q}^0 (t) = (q_1^0 (t), q_2^0 (t), ..., q_n^0 (t))^T$$

will describe some curve in the considered coordinate space. Let in the time $t_2$ system be in point $C$. Thus, the time interval from $t_1$ to $t_2$ the mechanical system will move along the curve $BC$. Some point on the curve $BC \ [t_1, \ t_2]$ corresponds to each time $t \in [t_1, \ t_2]$. The mechanical system is deformed under the influence of applied forces.

290
At any time $t$ one can determine the kinetic energy of the system $T(t)$, strain energy $U(t)$, and the work of $A(t)$ of external forces on the respective displacements; external forces also include non-conservative forces of resistance to motion (dissipative forces). If the system moved along other curve that connects points $B$ and $C$, then, other values of $T$, $U$ and $A$ would correspond to each time $t$.

In the case that the first approach (dynamic equilibrium equations are considered) is used in analysis of the mechanical system these equations are fulfilled in the points of the curve $q(t,0) = q^0(t)$, i.e. the vector sum of the inertial forces, elastic forces and external forces is zero. For points lying on other curves the equations of dynamic equilibrium are not, generally speaking, satisfied. One can distinguish true ("straight") path from the other ("misleadings") by the values $T(t)$, $U(t)$ and $A(t)$. Hamilton's principle states that on "direct" ways

$$\delta \int_{t_1}^{t_2} (T - U) dt + \int_{t_1}^{t_2} \delta A dt = 0. \quad (5.16)$$

Note that, proceeding from Hamilton principle, you can obtain the equations of motion that coincide completely with the equations obtained by the method of dynamic equilibrium or principle of possible displacements. Moreover, if any of the three considered principles is taken as initial, the other two can be obtained from it as a consequence.

In most mechanical systems kinetic energy is expressed through generalized coordinates and their first time derivatives, the potential energy – only through generalized coordinates, while the work of non-conservative forces on possible displacements caused by variation of generalized coordinates is a linear function of these variations. Just owing to this circumstance it is easy to switch from the integral equality (5.16) to a system of differential equations. Along with the equality (5.16) the following expression may be used to record the Hamilton principle:
\[ \delta \oint_{t_1}^{t_2} L dt + \oint_{t_1}^{t_2} \delta A dt = 0, \]  

(5.17)

where so-called Lagrange function (kinetic potential)

\[ L = L(t, y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n) = T - U \]

(5.18)

is the difference between kinetic and potential energy of the system.

W. Hamilton considered the case where the Lagrange function \( L \) does not depend explicitly on time \( t \), i.e. \( L = L(y_j, \dot{y}_j) \), corresponding to the case of stationary constraints. Investigations of W. Hamilton were summarized by M.V. Ostrogradsky in 1848 and V.F. Donkin in 1854 in a case, when Lagrange function depends explicitly on time \( t \), i.e. \( L = L(y_j, \dot{y}_j, t) \), that corresponds to nonstationary constraints. In this regard, the Hamilton principle is also called the principle of Hamilton-Ostrogradsky.

Note that it does not follow from the condition of the first variation of action function (\( \delta S = 0 \)) equality to zero that the action function (by Hamilton) \( S \) has extreme values. It is also necessary to examine the second variation \( \delta^2 S \).

Serres has shown that the second variation of the action for real motion with some restrictions imposed on integration limits is positive and, accordingly, the function \( S \) has a minimum. That is why the Hamilton principle is also called the principle of least action.

W. Hamilton defines the place of his principle of least action in the system of physical sciences: "Though the law of least action entered, therefore, in the number of the highest theories of physics, its claims to cosmological necessity based on economy in the universe are now usually discarded. Among other reasons that follows from the fact that the value which pretends to be saved is often wastefully spent in fact."

Due to the fact that the problem of searching minimum of the integral \( S \) is the task of the calculus of variations, the Hamilton principle belongs to variational principles of mechanics. It is an
integral principle because the system motion is studied in a finite period of time. Hamilton's principle is invariant with respect to the choice of coordinate system.

In accordance with the Hamilton-Ostrogradsky principle the system state is characterized by Lagrange variables $t, y_j, \dot{y}_j$ ($j=1,\ldots,n$), that is by point in time, as well as by position and velocities of the system points.

There is also another formulation of the Hamilton principle – in the form of Poincaré (principle of Hamilton-Poincaré) in which Hamilton variables $t, y_j, p_j$ ($j=1,\ldots,n$), where $p_j$ – summed pulses defined by equalities are used to characterize the system state

$$p_j = \frac{\partial L}{\partial \dot{y}_j}, \quad j=1,\ldots,n,$$

(5.19)

Hamilton variables can be expressed in terms of Lagrange variables and vice versa, and the system state can be characterized both by the values of Lagrange variables and values of Hamilton variables.

The mathematical recording of the Hamilton-Poincaré principle looks as:

$$\delta \int_{t_1}^{t_2} \left( \sum_{j=1}^{n} p_j \dot{y}_j - H \right) dt + \int_{t_1}^{t_2} \delta A dt = 0,$$

(5.20)

where the Hamilton function $H = H(y_j, p_j, t)$ is a result of the transition from Lagrange function $L(y_j, \dot{y}_j, t)$ using the Legendre transform and Donkin theorem

$$H(y_j, p_j, t) = \sum_{j=1}^{n} p_j \dot{y}_j - L(y_j, \dot{y}_j, t).$$

(5.21)

Generalized velocities designated by $\dot{y}_j$ are expressed in terms of Hamilton variables. Thereat positional coordinates $y_j$ and time $t$ are passive variables.
Note that since Legendre transformation is involutive, it is easy, following the same scheme, to revert to the Lagrange function and variables. The general scheme of mutual transformation is as follows:

<table>
<thead>
<tr>
<th><em>Old system</em></th>
<th><em>New system</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>Function: Lagrange’s, variable: velocity</td>
<td>Function: Hamilton’s, variable: impulses</td>
</tr>
</tbody>
</table>

Passive variables: position coordinates, time

Dual nature of transformation is reflected in the following operations:

1. Introduction of new variables

\[ p_i = \frac{\partial L}{\partial \dot{y}_i}, \quad \dot{y}_i = \frac{\partial H}{\partial p_i}, \]

2. Introduction of new functions

\[ H = \sum p_i \dot{y}_i - L, \quad L = \sum p_i \dot{y}_i - H, \]

3. Expression of new functions through new variables

\[ H = H(y_1, \ldots, y_n, p_1, \ldots, p_n, t), \quad L = L(y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n, t). \]

Thus, proceeding from Lagrange function \( L \) and using three sequential operations one can construct the Hamilton function \( H \). One can begin from Hamilton function \( H \) and construct, using three sequential operations, the Lagrange function \( L \).

Note also that the derivatives of the Laplace and Hamilton functions, in terms of passive variables, in accordance with Donkin theorem are connected by ratios

\[ \frac{\partial L}{\partial y_i} = -\frac{\partial H}{\partial \dot{y}_i}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \]

Thus, proceeding from Hamilton-Ostrogradsky principle (5.16) we can obtain the system of \( n \) ordinary differential equations of the second order relative to generalized coordinates \( y_1(t), y_2(t), \ldots, y_n(t) \), and using the Hamilton-Poincaré principle (5.20) we obtain a system of \( 2n \) first order differential equations relative to generalized
coordinates \( y_1(t), y_2(t), \ldots, y_n(t) \) and generalized impulses \( p_1(t), p_2(t), \ldots, p_n(t) \).

The last 22 years of his life Hamilton nearly entirely dedicated to design and development of calculus of quaternions and their practical application.

Hamilton died on September 2, 1865 at the age of 60. He is the author of 141 published works on various aspects of mathematics, optics and dynamics.

Thus were laid the foundation of Hamilton analytical mechanics that further became the basis of dynamics in terms of Hamilton-Jacobi, since the great German mathematician Jacobi (1804-1851) brilliantly developed, specified and considerably enriched Hamilton’s ideas in integration of differential equations of motion.

Karl Gustav Jacob Jacobi was born in 1804 in a family of Potsdam banker. He graduated from Berlin University and in 1825 defended his thesis. From 1826 he spent 17 years working in Königsberg.

The versatile mathematical work of Jacobi, his brilliant teaching talent, famous sarcasm that terrified opponents allowed him not only to widely influence his contemporaries, but also to create a scientific school. The unceasing striving for new, desire for changes were typical of Jacobi, he lacks calmness required to complete logically coherent theories. No wonder that Jacobi once said, "Gentlemen, we have no time for Gaussian rigour." Jacobi said of mathematics:
«Mathesis est scientia earum quae per se clara sunt» (Mathematics is one of those sciences that are clear themselves).


Thus, in the first period of formation of variational principles of mechanics, their development is essentially an integral part of the calculus of variations and problems of constructing analytical mechanics. The development of the calculus of variations gave mathematical methods of analytical mechanics, the development of the latter was one of the important reasons that led to the creation of variational calculus and thereafter continuously expanded the range of its problems.

References
Aleksandrov, P.S.(ed.) (1969), Problemy Gilberta [Hilbert problems], Nauka, Moscow, USSR.
Aleksyeyev, V.M., Tikhomirov, V.M. and Fomin, S.V. (1979), Optimalnoye upravleniye [Optimal control], Moscow, USSR.


Arnold, V.I. (1989), Huygens i Barrow, Newton i Hook – pervyie shagi matematicheskogo analiza i teorii katastrof, ot evolvent do kvazikristallov [Huygens and Barrow, Newton and Hook – first steps of mathematical analysis and theory of catastrophes, from evolvents to quasi-crystals], Nauka, Moscow, USSR.

Arnold, V.I. (1989), Matematicheskie metody klassicheskoi mekhaniki [Mathematical methods of classical mechanics], Nauka, Moscow, USSR.


Akhiezer, N.I. (1955), Lektsii po variatsionnomu ischisleniyu [Lectures in variational calculus, Gostekhizdat, Moscow, USSR.


Bogolyubov, A.N. (1986), *Geometriya i mekhanika v tvorchestve J.V. Poncelet* [Geometry and mechanics in crative work of Poncele], Issledovaniya po istorii fiziki I mekhaniki, Nauka, Moscow, USSR.


Vipper, Yu.F., (1875), *Semeistvo matematikov Bernulli* [The Bernoulli family of mathematicians], Moscow, Russia.


Gantmakher, F.R. (1966), *Lektii po analiticheskoj mekhanike* [Lecture in analytical mechanics], Moscow, USSR.

Gelfand, I.M. and Fomin, S.V. (1961), *Variatsionnoye ischislenie* [Variational calculus], Fizmatgiz, Moscow, USSR.

Gerts, G. (1959), *Printsipy mehaniki, izlozhennyie v novoi svyazi* [Principles of mechanics stated in new respect], AN SSSR, Moscow, USSR.

Gnedenko, B.V. (1952), *Mikhail Vasilievich Ostrogradsky* [Mikhail Vasilievich Ostrogradsky], GITTL, Moscow, USSR.

Gnedenko, B.V. and Pogrebysskiy, I.B. (1963), Mikhail Vasilievich Ostrogradsky, Izdatelstvo AN SSSR, Moscow, USSR.

Goldstein, G. (1957), *Klassicheskaya mekhanika* [Classical mechanics], Gostekhizdat, Moscow, USSR. Gantmakher, F.R. (1966), *Lektii po analiticheskoj mekhanike* [Lecture in analytical mechanics], Moscow, USSR.

Gelfand, I.M. and Fomin, S.V. (1961), *Variatsionnoye ischislenie* [Variational calculus], Fizmatgiz, Moscow, USSR.

Gerts, G. (1959), *Printsipy mekaniki, izlozhennyie v novoi svyazi* [Principles of mechanics stated in new respect], AN SSSR, Moscow, USSR.

Gnedenko, B.V. (1952), Mikhail Vasilievich Ostrogradsky, GITTL, Moscow, USSR.

298
Gnedenko, B.V. and Pogrebysskiy, I.B. (1963), Mikhail Vasilievich Ostrogradsky, Izdatelstvo AN SSSR, Moscow, USSR.

Goldstein, G. (1957), Klassicheskaya mekhanika [Classical mechanics], Gostekhizdat, Moscow, USSR.


Grigoryan, A.T. (1961), Ocherki po istorii mekhaniki v Rossii [Essays in the history of mechanics in Russia], Izdatelstvo AN SSSR, Moscow, USSR.

Grigoryan, A.T. (1964), Mikhail Vasilievich Ostrogradsky [Mikhail Vasilievich Ostrogradsky], Izdatelstvo AN SSSR, Moscow, USSR.

Grigoryan, A.T. (1974), Mekhanika ot antichnosti do nachikh dni [Mechanics from Antiquity to our days], Nauka, Moscow, USSR.

Grigoryan, A.T. (1979), Ocherki istorii mekhaniki v SSSR [Essays in the history of mechanics in the USSR], Russkiy yazyk, Moscow, USSR.


Grigoryan, A.T. and Fradlin, B.N. (1982), Istoriya mekhaniki tvyordogo tela [The history of mechanics of a solid body], Nauka, Moscow, USSR.

299

Huygens, H. (1935), Traktat o svete [Treatise on light], GTTI, Moscow-Leningrad, USSR.

Descartes, R. (1938), Geometriya. Z dodatkom vybranykh robit P. Ferma i lystuvannya Dekarta [Geometry. With adding selected works by P. Ferma and Desartes correspondence], GONTI, Moscow-Leningrad, USSR.

Demyanov, V.P. (1991), Rytsar tochnogo znaniya (P.L.Chebyshev) [A knight of exact knowledge (P.L.Chebyshev)], Znanie, Moscow, Russia.


Duvot, G. And Dions, J.-L. (1981), Neravenstva v mekhanike i fizike [Inequalities in mechanics and physics], Mir, Moscow, USSR.

Duem, P. (1903), Razvitije mekhaniki [Development of mechanics], Russia.

Evgurov, G.K. (1956), Zhuravsky Dmitriy Ivanovich (1821-1891), Collected articles: Uchonye i izobretateli zheleznodorozhnogo transporta, Gosudarstvennoye transporte zheleznodorozhnye izdatelstvo, Moscow, USSR.

Zhukovsky, N.E. (1950), Uchonye trudy M.V. Ostrogadskogo po mekhanike [Scientific works by M.V. Ostrogradsky in mechanics], Collected works, Vol.7, Moscow-Leningrad, USSR.

Idelson, N.I. (1975), Etyudy po istorii nebeshoi mekhaniki [Studies in the history of celestial mechanics], Nauka, Moscow. USSR.

Bogolyubov, A.N. and Shtokalo, I.Z. , Eds (1987), Istoriiya mekhaniki v Rossii [The history of mechanics in Russia], Naukova dumka, Kiev, USSR.

Grigoryan, A.T. and Pogrebyskky, I.B., Eds (1971), Istoriiya mekhaniki s drevneishikh vremyon do kontsa XVIII veka [The history of mechanics from the ancient times to the end of the 18th century], Nauka, Moscow, USSR.

Grigoryan, A.T. and Pogrebyskky, I.B., Eds (1972), Istoriiya mekhaniki s XVIII veka do serediny XX veka [The history of mechanics from the end of the 18th century to the middle 20th century], Nauka, Moscow, USSR.

Kepler, I. (1935), Novaya stereometriya vinnykh bochek [New stereometry of wine tuns], ONTI-GTTI, Moscow-Leningrad, USSR.

300
Klein, F. (1937), *Lektssel o razvitii matematiki v XIX stoletii* [Lecture on development of mathematics in the 19th century], ONTI, Moscow-Leningrad, USSR.


Kosmodemyansky, A.A. (1982), *Ocherki po istorii mekhaniki* [Essays in the history of mechanics], Nauka, Moscow, USSR.

Kotek, V.V. (1961), *Leonard Eiler* [Leonard Euler], Uchpedgiz, Moscow, USSR.


Krylov, A.N. (1943), *Nyuton i ego znachenie v mirvoi nauke (1643-1943)* [Newton and his significance in the world science (1643-1943)], Izdatelstvo akademii nauk SSSR, Moscow-Leningrad, USSR.

Kudryavtsev, P.S. (1943), *Isaak Nyuton, 1643-1943. K 300-letiyu so dnya rozhdeniya* [Isaac Newton, (1643-1943). On the 300th anniversary of his birth], A.Timiryazev, Uchpedgiz, Moscow, USSR.


Kudryavtsev, P.S. (1955), *Isaak Nyuton* [Isaac Newton], Uchpedgiz, Moscow, USSR.

Kudryavtsev, P.S. (1958), *Evangelista Torichelli – k 350-letiyu so dnya rozhdeniya* [Evangelista Torichelli – on the 350th anniversary of his birthday], Znanie, Moscow, USSR.


Kudryavtsev, P.S. (1974), *Kurs istorii fiziki* [Course of the history of physics], Prosveshchenie, Moscow, USSR.

301


Lavrentyev, M. and Lychestnik, L. (1935), *Kurs variatsionnogo ischisleniya* [Course of variational calculus], Gostekhizdat, Moscow, USSR.

Lagrange, J. (1950), *Analiticheskaya mekhanika, Tom 1* [Analytical mechanics, Vol. 1], GITTL, Moscow-Leningrad, USSR.

Lagrange, J. (1950), *Analiticheskaya mekhanika, Tom 2* [Analytical mechanics, Vol. 2], GITTL, Moscow-Leningrad, USSR.

Levi-Civita, T. and Amaldi, U. (1934), *Kurs teoreticheskoi mekhaniki. T.1, Ch. 1* [Course of theoretical mechanics. Vol.1, P. 1], ONTI, Moscow, USSR.

Levi-Civita, T. and Amaldi, U. (1962), *Kurs teoreticheskoi mekhaniki. T.1, Ch. 2* [Course of theoretical mechanics. Vol.1, P. 2], Moscow, USSR.

Leibnitz, H.V. (1982), *Sochineniya v 4 tomakh. T. 1*, [Collected works in 4 volumes, Vol. 1], Moscow, USSR.


Mach, E. (1909), *Mekhanika* [Mechanics], St Petersburg, Russia.


Mihkhlin, S.G. (1966), *Chislennaya realizatsiya variatsionnykh metodov* [Numerical realization of variational methods], Nauka, Moscow, USSR.

Mihkhlin, S.G. (1970), *Variatsionnyie metody v matematicheskoi fizike* [Variation methods in mathematical physics], Nauka, Moscow, USSR.

Newton, I. (1936), *Matematicheskie nachala naturalnoi filosofii* [Mathematical principles of natural philosophy], Transl. from Latin A.N. Krylov, Izd. AN SSSR, Moscow-Leningrad, USSR.


Ostrogradsky, M.V. (1959-1961), *Polnoye sobraniye trudov v dvukh tomakh* [Complete collected works in two volumes], Izd. AN Ukr.SSR, Kiev, USSR.

Ostrogradsky, M.V. (1968), *Izbrannyie trudy* [Selected works], AN SSSR, Moscow, USSR.

Polak, L.S. (editor) (1959), *Variasionnyie printsipy mekhaniki. Sbornik statei klassikov nauki* [Variational principles of mechanics. Collected papers of sciene classics], Fizmatgiz, Moscow, USSR.

Polak, L.S. (editor) (2010), *Variationnyie printsipy mekhaniki.* [Variational principles of mechanics], ed. 2, revised, Knizhnyi dom LIBORKOM, Moscow, Russia.


Poincare, A. (1910), *Nauka i method* [Science and method], St. Petersburg, Russia.


Poincare, A. (1983), *O nauke* [About science], Nauka, Moscow, USSR.

Rabinovich, I.M. (1975), *Voprosy teorii staticheskogo rascheta sooruzheniy s odnostoronnimi svyazyami* [Problems of the theory of static analysis of structures with unilateral constraints], Stroyizdat, Moscow, USSR.

Rakheyev, E.N. (1984), Dmitriy Ivanovich Zhuravsky, Nauka, Moscow, USSR.

Timoshenko, S.P. (1957), *Istoriya nauki o soprotivlenii materialov s kratkimi svedeniyma iz istorii teorii uprugosti i teorii sooruzheniy* [The history of science on strenght of materials with brief data from the history of theory of elasticity and theory of structures], Gosteeorizdat, Moscow, USSR.

303
Tyulina, I.A. (1979), *Istoriya i metodologiya mekhaniki* [The history and methodology of mechanics], Izd. Moskovskogo universiteta, Moscow, USSR.


Filonovich, S.R. (1988), *Sharl Kulon* [Charles Coulomb], Prosveshenie, Moscow, USSR.

Filonovich, S.R. (1990), *Tomas Young kak istorik nauki. Issledovaniya po istorii fiziki i mekhaniki* [Thomas Young as historian of science. Studies in the history of physics and mechanics], Nauka, Moscow, USSR.


Tslaf, L.Ya (1970), *Variatsionnye ischislenie i integralnye uravneniya* [Variational calculus and integral equation], GIMFL, Moscow, USSR.


Chebyshev, P.L. (1848), (1870), *Polnoye sobraniye sochineniy* [Collected works], Vol. 4, Moscow-Leningrad, USSR.

Chernov, S.N. (1935), *Leonard Euler i Akademiya nauk // Leonard Euler (1707-1783), Sbornik statei i materialov k 150-letiyu so dnya smerti* [Leonard Euler and Academy of Sciences // Leonard Euler (1707-1783). Collected papers and materials on the 150th anniversary of death], Izdatelstvo AN SSSR, Moscow-Leningrad, USSR.

Schmutzer, E. and Schutz, V. (1987), *Galileo Gallilei* [Gallileo Gallilei], Mir, Moscow, USSR.

Euler, L. (1934), *Metod nakhozhdeniya krivykh linii, obladayushchikh svoistvami maksimuma libo minimuma, ili resheniye izoperimetricheskoi zadachi, vzatoi v samom shirokom smysle* [Method of finding curves possessing properties of maximum or minimum, or solution of the isoperimetric problem, taken in the broadest sense], GTTI, Moscow-Leningrad, USSR.

304
Euler, L. (1959), *Dissertatsiya o printsipe naimenshego deistviya, s razborom vozrazheniy slavneishego prof. Koenig, proposed against this principle* (1753). *Sb. Variatsionnyie printsipy mekhaniki* [Thesis on principle of the least action with consideration of objections of the greatest prof. Koenig advanced against this principle (1753). Coll. Variational principles of mechanics], Ed by L.S. Pollak, Fizmatgiz, Moscow, USSR.

Euler, L. (1958), *Integralnoye ischislenie, v. 3* [Integral calculus, t. 3], Fizmatgiz, Moscow, USSR.


Elsgolts, L.E. (1958), *Variatsionnoye ischislenie* [Variational calculus], GITTL, Moscow, USSR.

Elsgolts, L.E. (1965), *Differentsialnyie uravneniya i variatsionnoye ischislenie* [Differential equations and Variational calculus], Nauka, Moscow, USSR.


Yaglom, I.M. (1977), *Feliks Klein i Sofus Li* [Felix Klein and Sofus Lie], Znanie, Moscow, USSR.

Jacobi, C. (1936), *Lektsii po dinamike* [Lecture on dynamics], ONTI, Moscow-Leningrad, USSR.

Young, L. (1974), *Lektsii po variatsionnomu ischisleniyu i teorii optimalnogo upravleniya* [Lectures on variational calculus and optimal control theory], Mir, Moscow, USSR.


Appell, P. (1900), *Traité de Mécanique, t. III.*


305

Castigliano, A. (1879), Theorie de l'équilibre des systemes elastiques, Turin.


Courtivron, O. de. (1749), Recherches de Statique et de Dynamique ou l'on donne un nouveau principe general pour la consideration des corps animés par des forces variables, suivant une loi quelconque, *Memoires de l'Academie des Sciences de Paris*.

D’Alembert, J.R. (1743), Traité de dynamique, dans lequel les lois de l'équilibre & du mouvement des corps sont réduites au plus petit nombre possible, David L'aîné, Paris, France.


De l'Hospital. (1696), Analyse des infiniment petits, pour l'intelligence des lignes courbes, Paris, France.

Euler, L. (1744), Methodus inveniendi lineas curvas maximis minimive proprietate gaudentes, Marc Michel Bousquet, France.


Föppl, A. (1892), Das Fachwerk im Raume, Leipzig, Germany.


Galileo Galilei. (1638), Discorsi e dimostrazioni matematiche intorno a due nuove scienze. In Leida, MDCXXXVIII.

Helmholtz, H. (1902), Dynamik continuallich verbreiteten Massen, Verlag von Johann Ambrosius Barth., Leipzig, Germany
Hertz, H. (1880), Über die Induktion rotierender Kugeln, Berlin, Germany.
Karman, T. (1910), Encyklopaedie der mathematischen Wissenschaften, T. VI.
Lagrange, J.L. (1788), Mécanique analytique, 1re éd., Paris, France.
Mach, E. (1883), Die Mechanik in ihrer Entwicklung, historisch-critisch dargestellt, Leipzig, Germany.
Maupertuis, P. (1740), La loi du repos., *Mem. de l'Acad. de Paris*.
Maupertuis, P. (1756), Oeuvres de Maupertuis, t. 4, Lyon, France.
Mobius, A. F. (1837), Lehrbuch der Statik (2 volumes).
Mohr, O. Z. (1874), Architek. u. Ing. Ver., Hannover, Germany.

307
Mohr, O. (1885), Ziviling.
Mohr, O. Abhandlungen aus dem Gebiete der technischen Mechanik
Ostrogradsky, M. (1848, 1958) Sur les integrales des equations generales de la
dynamique, Melanges de L’academie de St. Petersbourg, 6/18 oct 1848,
Selected works, AN SSSR, 1958.
Poincare, H. (1892), Les methodes nouvelles de la Mécanique celeste, t. 1. Paris,
France.
Spiess, O. (1948), Die Mathematiker Bernoulli, Basel, France.
Stevin, S. (1586), De Beghinselen der Weeghconst.
Trefftz, E. (1925), Ueber die Spanungsverteilung in tordierten Staben bei
teilweiser ueberschreitung der Fließgrenze, Zeitschrift für angewandte
Mathematik und Mechanik, Band 5.
Turner, M.J., Clough, R.W., Martin, H.C. and Topp, L.J. (1956), Stiffness and
805-824.
Truesdell, C. (1966), Six lectures on modern natural philosophy, Springer-
Verlag, New York, USA.
Truesdell, C.A. (1968), Essays in the history of mechanics, Springer Verlag,
Berlin, Germany.
Truesdell, C. (1972), Leonard Euler, supreme geomter (1707-1783). Studies in
XVIIIth Century Culture, v. 2. Case Western Reserve Univ. Press.
Truesdell, C. (1980), The Tragicomical History of Classical Thermodynamics,
1822-1854, Springer-Verlag.
Varignon, P. (1725), Nouvelle mécanique, 2, Paris, France.
Zienkiewicz, O.C. and Cheung, Y.K. (1967), The finite element method in
engineering science, McGraw-Hill,London, UK [Russian translation,
(1974), Nedra, Moscow, USSR].
Essay 6

BASIC VARIATIONAL PRINCIPLES AND FUNCTIONALS OF STRUCTURAL MECHANICS
...science has its own esthetics, and the beauty of logic orderliness of variational principles of mechanics cannot help but surprise...

L. Polak
6.1. Variational principles of mechanics of solids

Navier’s memoir, presented to the Paris Academy of Sciences in 1821, was published in 1827 [Navier, 1827] (shortened in 1823) [Navier, 1823]. Proceeding from Newton’s conception on the matter structure, developed by Boscovich [Boscovich, 1763], Navier has written an expression for a sum of all forces affecting a molecule at small displacement; and using the methods of variational calculus he obtained differential equations of equilibrium and motion in displacements, as well as a boundary condition. Just this work had probably made A. Cauchy to expound his famous principle of stresses in the memoir presented to the Paris Academy of Sciences in 1822, which short content was published as an article [Cauchy, 1823] in 1823. The conception of elastic potentials in general form was advanced in 1839 by G. Green [Green, 1839], who derived the equation of elasticity theory on the basis of the principle of energy conservation. Lord Kelvin has proved the existence of the function of elastic potential on the basis of the first and second laws of thermodynamics.

It is considered in classical structural mechanics of bar systems [Rabinovich, 1954] that the principle of possible displacements for deformed bodies was first used in 1833 by Poisson. To calculate the system of the arbitrary number of bars with their ends hinged to arbitrary number of nodes Poisson has obtained such an equation which represents a principle of possible displacements for deformed bodies:

\[ \sum Q_m \delta_m - \sum S \Delta \bar{\delta} = 0, \]

where \( Q_m \) – external forces, \( \delta_m \) – possible displacements of nodes in directions of these forces, \( S \) – forces in bars, \( \Delta \bar{\delta} \) – continuation of bars corresponding to displacements \( \delta_m \). And though here the system elasticity properties were not allowed for, the work of Poisson is considered as a stage in preparation to further works.
It should be noted that L. Euler, using the data from the letter of Daniel Bernoulli of October 22, 1742, applied variational calculus to solve the problem that "when the value of the expression \( \int \left( \frac{1}{\rho^2} \right) \, ds \) is the least" and examined the displacement of a bended cantilever beam, certainly, without using a physical notion of full potential energy of the system. It is of interest that the problem on brachistochrone was solved in 1696, that was the beginning of variational calculus (the theory of functionals extrema). The symbol of variation was introduced by Lagrange, though the term variation itself was introduced by L. Euler much later, and the term functional – by J. Hadamard only in 1903.

In 1833 J. Fourier had first introduced the notion of constraints and obtained Fourier inequality. In 1851 Silvester introduced the term invariant.

The Lagrange and Castigliano theorems have been formulated in the course of time. Lagrange formula (first Castigliano formula), like the principle of possible displacements itself, is valid for any (linear or nonlinear) deformed system. The nature of the Lagrange formula is analogous to that of G. Green formula in the theory of elasticity. This formula is probably connected with the name of J.-L. Lagrange in the sense that it follows from the Lagrange variational principle.
However, it was obtained by Carlo Alberto Castigliano (1847-1884), and, since there exists another symmetrical formula, this one is sometimes also called the first Castigliano formula.

In 1884 E. Winkler, a founder of the Berlin school of the theory of structures, wrote, both with animation and bitterness, in the obituary of C.A. Castigliano: “Theory of structures was to a certain extent founded by Italians such as Galileo, Marchetti, Fabri, Grandi, etc; but said theory has made important progress in recent times as a result of the needs brought about by the introduction of railways, and the Italians are again playing a prominent role in this progress. The more recent works of Allievi, Biadego, Canevazzi, Ceradini, Clericetti, Cremona, Favaro, Favero, Figari, Guidi, Jung, Modigliani, Saviotti and Sayno, to name but a few; outstanding among these works are those of Castigliano. If we Germans also wish to claim that our efforts in these areas are also worthy of note, then we must admit that we have learned much from our Italian colleagues and that, regretfully, language barriers still prevent a faster dissemination of their theories”.

C.A. Castigliano studied elastic systems, using the theorem on the minimum of deformation work, in his graduation thesis. It soon appeared that this theorem coincided in many respects with the principle of the least work $\Pi = \min$, discovered by Luigi Federico Menabrea even in 1857. Indignant at contemptuous attitude, Menabrea issued in 1875 a paper, where he asserted his priority. Several months later Castigliano, in response to this paper, has published the essay of 150 pages entitled «Nuova teoria intorno all'equilibrio dei sistemi elasticiti» (New Theory of Equilibrium of Elastic Systems), where he
had left behind L.F. Menabrea and formulated the essence of his major work, which soon appeared in 1879.

The major work by C.A. Castigliano is based on three statements concerning deformation energy: “If we express the function of internal work of a body or elastic system through relative displacements of the points of application of external forces, the function displacement derivatives will give the values of the corresponding forces”:

$$\frac{\partial \Pi (..., \delta_k, ...)}{\partial \delta_k} = F_k$$

The first Castigliano theorem had been used earlier to physical problems by G. Green. This theorem translation for the problems on the theory of structures was a true creation by C.A. Castigliano, formulated by him in 1873 in his graduation thesis.

The second Castigliano theorem is formulated as follows: “The internal work of a body or elastic system being expressed as a function of external forces, this expression derivative, concerning one of the forces, will give a relative displacement of the point of force application”:

$$\frac{\partial \Pi (..., F_k, ...)}{\partial F_k} = \delta_k$$

The first Castigliano theorem may be obtained from this expression: “Stresses, which appear between the body or system parts after deformation, are such that the work of internal forces is minimal, hence follow equations, which express the equilibrium of forces, applied to each part”. This theorem corresponds to the Menabrea principle, which Castigliano refers explicitly to in the Introduction, but he adds that he has given its strict proof in his graduation thesis of 1873.

In the Introduction to his main work C.A. Castigliano states: “that is a given book, which embraces completely the theory of elastic forces in structures, ... is mainly based on the theorems on derivatives of the internal work”. Thus he has first introduced a principle of energy into the theory of structures.
Though some of the above results were obtained independently by L.F. Menabrea, as well as by English scientist James Henry Cotterill (1836-1922) the assignation of the name of Castigliano is rather historically justified, since it was he who has solved, using his own results, the great number of problem, and has created the apparatus for operation analysis.

E. Winkler has attracted the attention of his German colleagues to the major work of C.A. Castigliano *Theory of Equilibrium of Elastic Systems and Its Application*.

It should be noted that Italian engineer L.F. Menabrea (1809-1896) was one of the first scientists, which tried to formulate the principle called now the principle of possible force (stress) changes. His first work dedicated to this problem «Principio generale per determinare le tensioni e le pressioni in un sistema elastico» (General Principle for Determining Tensions and Pressure in the Elastic System) was presented at the seminar of the Academy of Sciences in Turin and published in 1857.

![George Green (1793–1841)  Carlo Alberto Castigliano (1847–1884)  Luigi Federico Menabrea (1809–1896)](image)

There were some discrepancies in the work. Firstly, L.F. Menabrea did not understand that this principle formulation rather includes a certain mathematical value than potential deformation energy; this value was later called complementary energy. Secondly, real displacements did not appear in the principle formulated by Menabrea (as it had to be) but their variations and, at last, he had not emphasized
that equilibrium equations were to be satisfied by stress variations, i.e.
he had not emphasized a statical possibility of stress variations.

Great and long-term discussion appeared in conection with
Menabrea’s works and above inaccuracies. One of this discussion
participants Joseph Louis François Bertrand (1822-1900) in his letter
to L.F. Meabrea in 1869 informed the latter about a necessity of
introducing changes into the principle \( U^\text{comp} \) instead of \( U, \Delta_i \) instead
of \( \delta \Delta_i \) and statical possibility \( \delta P_i \). In 1870 L.F. Menabrea had
published a paper, where he took in to account all notes of J.L.F.
Bertrand (from a fragment of Menabrea’s letter, which they had jointly
published in Proceedings of the Academy of Sciences in Turin on May
1, 1870, P. 702). Thus the present formulation of possible stress belongs
to Bertrand. But since, in spite of availability of errors in the initial
formulation of the principle, L.F. Menabrea used it in solving numerous
problems and did no mistakes (since he considered linear system for
which \( U^\text{comp} = U \), variations of forces were taken as statically possible
without signing this, and \( \delta \Delta_i \) was actually understood as \( \Delta_i \)) the name
of L.F. Menabrea is sometimes preserved in the principle denomination.

Independent of J.L.F. Bertrand the principle of additional elastic
potential (complementary elastic energy) was proposed by German
scientist Wilhelm Fränkel (1841-1895) in respect of the spatial problem
of the elasticity theory in the paper «Das Prinzip der kleinsten Arbeit
der inneren Kräfte elastischer Systeme und seine Anwendung auf die
Lösung baustatischer Aufgaben» (Principle of the Least Work of
Internal Forces of Elastic Systems and Its Application to Solving the
Problems of Statics) [Fränkel, 1882].

In 1921 German scientist Oscar Domke (1874-1945) studied the
principle of possible change of forces from the viewpoint of
thermodynamics [Domke, 1921] and established the extreme principle
for adiabatic elastic systems: all external force and temperature make
the value \( u^\text{comp} - \sum_i P_i \Delta_i - \sum TS \) minimal, i.e.
\[ \delta_{(Q,T)} \left[ u^{\text{comp}} - \sum_i P_i \Delta_i - \sum TS \right] = 0, \]

where \( T \) – absolute temperature, \( S \) – entropy.

Use of the theorem of the works to calculate displacements in the elastic systems was started by James Clerk Maxwell and Christian Otto More. J.C. Maxwell in the work of 1864, and O. More in a series of papers of 1874-1885 obtained a well-known formula for determining displacements in elastic truss with preset internal forces, which made it possible to analyze statically indeterminate structures.

In 1882 Mattias Koenen applied to general theorem of the work, successfully used by O. More in the theory of brace bar systems, and used it in the form of the principle of possible changes of the stress state for calculation of displacements in statically definable beams and reactions of supports in a continuous beam. In so doing he has first formulated the equation of work of a single force in a form of the known integral of the product

\[ \delta_i = \int \frac{M_i M_j}{EI} dx. \]

Friedel Hartmann has studied validity of Castigliano theorems in his monograph *Mathematical Principles of Structural Mechanics* (1985). Proceeding from the embedding theorem by S. L. Sobolev he obtained inequality \( m-i > n/2 \), where \( m \) is the order of derived
functions, entering in the expression of energy, \( i \) – singularity index, and \( n \) – continuum dimension. Castigliano theorems are valid only if this inequality is fulfilled. If concentrated forces are applied to the beam, singularity index \( i=0 \). Take into account that \( m=2 \) (other deflexion derivatives enter in the expression of energy) and as a result of substitution \( 2 - 0 > 1/2 \). Thus, the above inequality is satisfied, and Castigliano theorems may be used. The inequality is not satisfied for two-dimensional \( (n=2) \) and three-dimensional bodies \( (n=3) \), which are loaded by concentrated forces, since \( m=1 \leq n/2 \). If the concentrated moment \( (i=1) \) acts, Castigliano theorems are also used only for beams and plates, which deformation takes place in accordance with the Kirchhoff-Lave hypothesis. When applying loads with corresponding solutions with a higher singularity index, the Castigliano theorems cannot be use in any case [Hartmann, 1985].

S.L. Sobolev (1908–1989) – Soviet mathematician and mechanicist, one of the most famous mathematicians of the 20\(^{th}\) century, who had made the basic contribution to the present science, started in his fundamental researches a number of new scientific trends in modern mathematics.

A classical phase (1875-1900) of transformation of the theory of structures into the fundamental discipline of technical sciences of civil building consisted in the struggle around its theoretical substantiation. As a result the method of forces created by Henrich Franz Bernhard Müller-Breslau and his students acquired its current form. H.F.B. Müller-Breslau had considerably extended his journal papers in the theory of statically undefinable trusses, published between 1882 and 1885, in his book in material resistance and theory of structures (1886), based on the principle of possible displacements and deformation work. In this monograph the problems of analysis of structural and technical peculiarities of carrying systems, arising from everyday building, were subject to processing based on a single theoretical basis of the principle of possible displacements (in a form of the principle of virtual forces), second Castigliano theorem (based on the principle of energy) and the Menabrea principle. This work, which
has run into several editions (1886, 1893, 1904, 1913, 1924), not only completed the period of formation of such a discipline as the theory of structures, started by the work of L. Navier «Résumé des Leçons» [Navier, 1826], but also allowed the classical theory of structures to substitute statics and material resistance. Strict formulation of the method of forces for trusses in its modern structure and form was the heart of this synthesis.

In the first edition H.F.B. Müller-Breslau introduces a notion of statically definable basic system and a single force $X_k = 1$, affecting this system. Following the idea of O.Mohr and expanding its effect for embracing the bending structures H.F.B. Müller-Breslau allows this single force to make work on true displacements (i.e. over a given system with $n$ steps of static indeterminedness) and thus obtains $n$ equations of elasticity; he obtains the same equations through Menabrea principle and superposition of equations for the internal forces. Almost all statically indeterminable problems were solved by H.F.B. Müller-Breslau with the use of the Menabrea principle. But when determining the influence lines for statical indeterminedness in a case of movable load $P_m$ he takes directly the principle of virtual forces and using Maxwell theorem $\delta_{mn} = \delta_{nm}$, which H.F.B. Müller-Breslau has generalized for the turns, expresses the first version of the method of forces [Müller-Breslau, 1886, pp. 138-140].

Immediately after the publication of «Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktion» (Modern Methods of Strength of Materials and Theory of Structures) O. Mohr published polemics, formulated in categorical terms, in the journal Zivilingenieur, which opposed the concept of "idealized strain" of
H.F.B. Müller-Breslau, which he distributed for special cases of load in the form of thermal effects and shifts of supports; one of his criticisms was the fact that "a number of other designations, such as working principle, principle of possible work, principle of possible displacements were used for the principle of possible velocities by defenders of newest methods» [Mohr, 1886, p. 398]. H.F.B. Müller-Breslau answered this consistent objection in 1892, symbolically separating the actual condition of displacement and causal condition of force from the theoretical condition of force [Müller-Breslau, 1892, p. 11 September]. The principle of virtual forces has first obtained the right to exist, regardless of the principle of possible displacements at the level of small shears, not in the name but the apparatus of equations of classical theory of structures.

While A. Mohr discussed the validity of Castigliano theorem for justification of the classical theory of structures, H.F.B. Müller-Breslau completed the classical theory of structures expanding successively the expression of deformation energy for elastic brace systems. That is why the doctrine of energy has become the dominant one in theory and practice about 1900. However, in the first decade of the 20th century Weingarten and Mertence tried to eliminate the domination of Castigliano theorems. The debates that accompanied it
resembled an old dispute between O. Mohr and H.F.B. Muller-Breslau, which largely concerned issues of priority. But by 1910 the debate was completed by Veirauch in favor of Castigliano theorems in the classical structural theory.

The development of structural mechanics in the period associated with the names of B.P. Clapeyron, G. Lame, J. Maxwell O. Mohr, J.V. Rayleigh, V.L. Kirpichov, S.P. Timoshenko, A. Lyav, Sophie Germain and others. Two of the most significant scientific achievements of Benoit Paul Emile Clapeyron are, likely, *Abhandlung über die bewegende Kraft der Wärme* (Treatise on the Motive Power of Heat) [Clapeyron, 1926] and the theorem that bears his name in the theory of elasticity. Giving Clapeyron’s name to this theorem is attributed to G. Lame, who in 1852 presented Clapeyron’s theorem for the general case of spatial elastic continuum in the first monograph on the theory of elasticity [Lamé, 1852, p. 80-92]. Both works of B.P. Clapeyron had a decisive effect on fundamental scientific and technological disciplines – structural theory and applied thermodynamics, which began forming from 1820.

Publication of the Clapeyron theorem on the work of elastic forces (1852) has marked the beginning of a new era of energy trend in the elasticity theory and structural mechanics. Clapeyron formula establishes the relationship between the work of external forces and potential energy of deformation. It is crucial that this dependence is valid for the actual state of the system, allows formulating different variational principles of mechanics by varying certain components.

Though the basic principles of thermodynamics, which were formulated before 1850 by Sadi Carnot (1796-1832), J.P. Joule (1818-1889), R. Mayer (1814-1879), H. Helmholtz (1821-1894), R. Clausis (1822-1888) and W. Thomson (1824-1907), D.U. Gibbs (1839 1903), were attributed to the most important discoveries of the 19th century by many well-known contemporaries; only decades later, they took a significant influence on the theoretical foundations of structural theory and applied thermodynamics at the time of their formation, which lasted three-quarters of a century (1825-1900).
Their formation was the result of creation of the steam engine, not as an invention to achieve a specific goal, but rather as a "universal motor for industry" [Marx, 1979, p. 398]. Steam locomotive has brought industrial revolution to the farthest neglected corners of the continent.

The fundamental variational principle of thermodynamics is the Gibbs principle of thermodynamic equilibrium.

For $k$-component $r$-phase system at a constant internal energy $U$ of the volume $V$ and the number of moles of components the condition of thermodynamic equilibrium is that at all possible changes in the parameters of the state the entropy of the system $S$ remains invariable or decreases. In other words, the entropy of the isolated system at thermodynamic equilibrium has a conventional maximum

$$(\delta S)_{U,V} \leq 0.$$ 

A sign of equality takes place when reversible processes proceed in the system, a sign of inequality – irreversible process (in the case of isolated system).

The principle of equilibrium can also be expressed through thermodynamic potentials – internal energy $U$, enthalpy $H$, Gibbs energy $G$, Helmholtz energy $F$ – under conditions that are characterized by consistently relevant state parameters. Conventional minimum of thermodynamic potentials corresponds to thermodynamic equilibrium.

$$(\delta U)_{S,V} \geq 0; (\delta H)_{P,S} \geq 0; (\delta G)_{V,T} \geq 0; (\delta F)_{V,T} \geq 0,$$

where $U$ – the internal energy, $S$ – entropy, $H=U+PV$ – enthalpy, $G=H-TS$ – isobar-isothermal potential (Gibbs free energy, free enthalpy); $F=U-TS$ – isochore-isothermal potential (Helmholtz free energy).

The system transition from one state of thermodynamic equilibrium to another can occur through a sequence of states, each of which is also a state of thermodynamic equilibrium. This means
that the parameters of the system state throughout the transition process differ infinitely little from their values at thermodynamic equilibrium. That is the equilibrium (quasi-static) process. In fact, the processes of transition are always non-equilibrium, they are studied by chemical thermodynamics [Bazarov, 2010], [Tyulyna, 1979], [Prigozhin, Defey, 1966].

Implementation of energy doctrine in the structural theory – a term introduced by the founder of physical chemistry, Wilhelm Ostwald (1853-1932), around 1900 – is no more than a projection of a real steam engine on scientific and technical model of a girder frame, which was made on the basis of energy approach by James Clerk Maxwell [Maxwell, 1864]. This becomes clear, when we compare the volume-pressure diagram of a heat engine (Fig. 6.1, a) with the force-displacement diagram of linear-elastic pass-through bar structures (Fig. 6.1, b). In both cases the area of outlined region determines energy of the corresponding technical artifact expressed in the form of mechanical work. B.P. Clapeyron has formulated the mathematical principles of the both diagrams.

![Diagram](image_url)

**Fig. 6.1.** a) Pressure-volume diagram by Clapeyron and b) Diagram of deformation dependence on stresses of one-dimensional body that obeys Hooke’s law.
The equation which corresponds to the solution of the problem on deformation of elastic hinged trusses on the basis of the principle of virtual forces was used by J.K. Maxwell to obtain the ratio $\delta_k = \delta_{ki}$ [Maxwell, 1864, p. 297], which in 1886 H. Müller-Breslau called the Maxwell theorem, as a tribute to this amazing physicist.

A special deriving of the Maxwell equation of Clapeyron’s theorem in a few words and without diagrams, played a great role in the dispute between Christian Otto Mohr and Franz Heinrich Bernhard Müller-Breslau on the theoretical substantiation of the classical structural theory, which was continued during the 1880’s.

After solving the problem, which concerned the deformation of statically determinable girder frame, J.K. Maxwell turned to analysis of statically undeterminable trusses. For this purpose, he chose a statically determinable main system with $s\cdot n$ rods and recorded the expression for the total force $S^{(n)}_j$ in the rod $j$ of statically undeterminable system that has $n$ spare rods

$$S^{(n)}_j = S^{(0)}_{j,0} + \sum_{k=1}^{n} S^{(0)}_{j,k} \cdot X_k.$$ 

Undoubtedly, the two-volume work by John William Strett, the third Baron Rayleigh The Theory of Sound [Rayleigh, 1877-78] belongs to the library of classic physics and is the first book devoted to acoustics. In his review of the first volume of the book for the journal Nature, H. Helmholtz writes: "The author will deserve the eternal gratitude of all, who study physics and mathematics, if he will continue his work in the same spirit in which he began the first volume. Because of its successful systematic organization of the entire volume of the most difficult problems of acoustics, the author made it possible to study this subject in much easier way than before" (cited by [Rayleigh, 1879 (preface by German translator)]. At the suggestion of G. Helmholtz, "a chancellor of German physics", two-volume work by J.W. Rayleigh was immediately translated into German [Rayleigh, 1879, 1880]. The second, enlarged and augmented edition in English
appeared in 1894 and 1896 [Rayleigh, 1894, 1896]; the same edition was reprinted unchanged by Dover publications in 1945.

Interestingly, John William Strett (Lord Rayleigh) was awarded the Nobel Prize in Physics in 1904 "for studying densities of most commonly used gases and for argon discovery in the course of these investigations."

But what a classic of acoustics has to do with the classical theory of structures? A look at the first volume of The Theory of Sound gives us a hint. After J.W. Raleigh considers fluctuations of all systems in general in the first four chapters, he examines vibrations of cables, rods, plates and membranes; in the second part of the volume he examines vibrations of membranes [Rayleigh, 1894, p. 395-432] and electrical oscillations [Rayleigh, 1894, p. 433-474]. His second volume is devoted to aerodynamic problems of acoustics.

The concept of energy heads the list in the acoustics of J.W. Rayleigh. For example, in the third section, General Oscillatory Systems [Rayleigh, 1894, p. 91-169], he starts with potential energy $P$ and kinetic energy $T$ and generalizes reciprocity theorem of elastic vibration systems, formulated by Betty E. in 1872, using the concept of generalized forces $F_k$ and corresponding generalized coordinates of displacements $\delta_k$ and d'Alembert principle: "If force of harmonic type with preset amplitude and period affects the system at the point $P$, then the displacement that occurs at another point $Q$, will take the same amplitude and phase, as the displacement at point $P$, if force was applied to $Q$" (cited by [Timoshenko, 1957, p. 383]). In his book "Redundant Unknowns in Structural Mechanics [Kirpichov, 1903, 1934] V.L. Kirpichov translates a problem of oscillations mathematically formulated by J.W. Rayleigh into the language of
power principle and formalism of J.-L. Lagrange on generalized coordinates and forces – to the level of structural analysis.

J.W. Rayleigh is a founder of the principle [Rayleigh, 1894, p. 109-112], which states that in a closed, in the sense of thermodynamics, system in which energy conservation law has the form

$$\Pi = T,$$

the first natural frequency $\omega_1^2$ of the system can be calculated as

$$\omega_1^2 = \frac{\Pi}{T/\omega_1^2}.$$  \hspace{1cm} (6.1)

The equation (6.2) is known as Rayleigh frequency expression. For beams with length $l$ on two supports with constant mass per unit length $\mu$ and bending rigidity $EI$, with the energy of deformation

$$\Pi = \frac{EI}{2} \cdot \int_0^l \left(\frac{d^2 \bar{w}(x)}{dx^2}\right)^2 \ dx$$  \hspace{1cm} (6.3)

and kinetic energy

$$T = \frac{\omega_1^2}{2} l \int_0^l \bar{w}^2(x) \ dx$$  \hspace{1cm} (6.4)

natural frequency derived from the law of energy conservation (6.1) is

$$\omega_1^2 = \frac{\frac{EI}{2} \cdot \int_0^l \left(\frac{d^2 \bar{w}(x)}{dx^2}\right)^2 \ dx}{\frac{\mu}{2} \cdot \int_0^l \bar{w}^2(x) \ dx}.$$  \hspace{1cm} (6.5)

The expression (6.5) is a particular form of the Rayleigh frequency expression (6.2), resulting from the comparison of denominator in (6.4) and (6.2). If we substitute a static deformation curve for oscillations form $\bar{w}(x)$ in (6.3) and (6.4), then we get a useful approximation

$$\omega_1^2 \approx \frac{\frac{EI}{l} \int_0^l \left(\frac{d^2 w(x)}{dx^2}\right)^2 \ dx}{\mu \cdot \int_0^l w^2(x) \ dx}.$$  \hspace{1cm} (6.6)

326
So J.W. Rayleigh has succeeded in determining the first natural frequency of oscillatory systems with energy conservation law (6.1) without solving the corresponding differential equation. Later, Walter Ritz further developed the Rayleigh method in the framework of variational calculus and formed a general approximate method (1909) – the Rayleigh-Ritz method. Ritz method for direct solution of variational problems, not only provides mathematical physics, but also applied mathematics and the theory of structures with an approximate apparatus capable of elegant solving the problems of elastostatics and elastokinetics, such as the calculation of critical load for the rod of variable rigidity.

Victor L. Kirpichov completed the period of formation of the discipline of structural theory in Russia with his book *Redundant Unknowns in Structural Mechanics* (1903), which contains only 140 pages. It explains the whole theory of statistically indeterminate strut frames in an extremely simple manner of presentation. So he and Müller-Breslau can be considered the figures which have summarized the classical theory of structures.

Like J.W. Rayleigh, V.L. Kirpichov based his work on the potential energy $I\!I$ (strain energy) and introduced the concept of generalized forces $F_k$ and the corresponding generalized displacements $\delta_k$. For a system of $n$-dimensional degree of freedom and for a special event associated with the time independence of $I\!I$ (conservative mechanical system), V.L. Kirpichov formulates the Lagrange equation:

$$ \sum_{k=1}^{n} \left( F_k - \frac{\partial I\!I}{\partial \delta_k} \right) d\delta_k = 0. \quad (6.7) $$

V.L. Kirpichov obtained the Lagrange equation (6.7) intended for
statics in the principle of possible displacements by comparing coefficients. Since \( d\delta_k \neq 0 \) the expression in brackets in (6.7) should disappear, that is the equality holds
\[
\frac{\partial \Pi}{\partial \delta_k} = F_k. \tag{6.8}
\]

The equation (6.8) is nothing but the Castigliano first theorem. According to V.L. Kirpichov generalized forces are not only forces in a narrow sense, but their various combinations that also include, for example, moments. Generalized displacements also are not only displacements in a narrow sense, but are complex displacements that also include turning angles. For V.L. Kyrpychov all these are special cases of generalized forces and generalized displacements. In the first half of consolidation period of the structural theory (1900-1950) this has led to the displacement method.

The method of forces developed parallel with the displacement method; the method of forces did not follow through the strain energy \( \Pi \), but through complementary deformation energy \( \Pi^{\text{comp}} \) (\( \Pi = \Pi^{\text{comp}} \) in the case of linear-elastic behavior of the material) and the principle of virtual forces. V.L. Kirpicheo develops this approach to design of statically undeterminable frames using the second Castigliano theorem and Menabrea principle in sections 6 and 7 of his book. In this case, he based his conclusions on the conception of generalized forces and displacements. In doing so, he admits the equality between the strain energy \( \Pi \) and complementary strain energy \( \Pi^{\text{comp}} \), denoting the both by the letter \( U \). The description with the use of generalized forces and displacements plus restrictions on linear-elastic behavior of the material provided work of Kyrpychov with limit clarity and accessibility.

Besides the Castigliano theorems, V.L. Kirpichov has also derived the reciprocity theorem. The structure of his theory of statically indeterminate trusses, based exclusively on the energy principle, conception of generalized forces and generalized displacements and on the Lagrange equation, largely attracts attention due to its versatility and concise form, which clarity remained unsurpassed for many years. For example, the section devoted to the lines of influence becomes particularly clear owing to his reciprocity theorem ([Timoshenko, 1957, p. 384]), which in its most general form was certainly proved by Rayleigh. Kirpichov achieved common
and consistent formulation of the theory of statically undeterminable, linear-elastic trusses that is not inherent in either the energy or kinematic doctrine in structural theory; thus he leaves the door open both for the method of forces and for the method of displacements. The total base under dual structure of the theory of structures, so clearly presented in the books by V.L. Kyrpychov foresees the way, which continuous innovations in the theory of structures of the first half of the 20th century will follow.

![Benois Paul Emile Clapeyron](image1) ![John Strutt, 3rd Baron Rayleigh](image2) ![Walter Ritz](image3) ![Viktor Lyovich Kirpichov](image4)

*The Theory of Sound* by J.W. Rayleigh had great influence on V.L. Kirpichov; and in his introduction, he recommends this work those his readers, who are interested in the theory of structures. That is why we should be grateful to V.L. Kirpichov because he made the Rayleigh method known, first in Russia and then in other countries. S.P. Timoshenko was one of the well-known students of V.L. Kirpichov. However, the effect of successful adaptation (made by V.L. Kirpichov) of the methods from *The Theory of Sound* by J.W. Rayleigh to the theory of statically indeterminate systems in the time frame was inferior to the influence of the Berlin school of structural mechanics, which played a dominant role in the formation of the theory in the first half of consolidation period of the structural theory.

In 1892 O.M. Lyapunov published the work *General Problem of the Stability of Motion*, which gave a rigorous mathematical formulation of stability problems of mechanical systems with a finite number of degrees of freedom. In 1932 M.G. Chetaev formulated and proved theorems on motion instability. In 1937 M.M. Krylov and N.N. Bogolyubov issued a monograph *Introduction to Nonlinear Mechanics*. A significant contribution to the development of dynamic stability of structures was made by V.V. Bolotin and his disciples.
Only in the 30’s of the past century the methods of design of statically indeterminate rod systems gained complete form; methods of forces, displacements and a mixed method and their various modifications were distinguished. V.L. Kirpichov, N.S. Streletsyky, A.A. Gvozdiov, P.L. Pasternak, I.M. Rabinovich, N.I. Bezukhov and others made great contribution to the process of their formation.

6.2. Principles of Lagrange and Castigliano

If other conditions, besides the additional condition \( M = EI\kappa \), have been taken, namely,

\[
\kappa = -\frac{d^2 w}{dx^2}, \quad \frac{d^2 M}{dx^2} + q = 0,
\]

\[
w|_{a_2}^{b_2} = \bar{w}|_{a_2}^{b_2}, \quad \bar{w}|_{a_2}^{b_2} = \bar{w}|_{a_2}^{b_2},
\]

we get the Lagrange and Castigliano functionals which depend only on one variable \( w \) or \( M \) [Bazhenov, 2014]

\[
\Pi^L(w) = \frac{1}{2} \int_a^b EI \left( \frac{d^2 w}{dx^2} \right)^2 dx - \int_a^b qw dx - M\bar{w}|_{a_2}^{b_2} + \bar{M}w|_{a_2}^{b_2},
\]

\[
\Pi^C(M) = -\frac{1}{2} \int_a^b \frac{M^2}{EI} dx + \bar{w}M|_{a_2}^{b_2} - M\bar{w}|_{a_2}^{b_2}.
\]

So we have a couple of dual problems of variational calculus that meet the variational principles of Lagrange and Castigliano

\[
\delta \Pi^L (w) = 0 \quad \text{with the additional condition} \quad \delta \Pi^K (M) = 0
\]

\[
\text{with the additional condition} \quad \delta \Pi^{\kappa} (w) = 0.
\]

In expanded form the indicated variational problems are as follows:

\[
\delta \Pi^L (w) = \int_a^b EIw^\kappa \delta w^\kappa dx - \int_a^b q\delta w dx - M\delta w|_{a_2}^{b_2} + \bar{M}\delta w|_{a_2}^{b_2} = 0,
\]

\[
\text{additional conditions:} \quad \bar{M}\delta w|_{a_2}^{b_2} + \bar{M}\delta w|_{a_2}^{b_2} = 0,
\]

\[
\delta \Pi^C(M) = -\int_a^b \frac{M}{EI} \delta M dx + \bar{w}M|_{a_2}^{b_2} - M\bar{w}|_{a_2}^{b_2},
\]

\[
\text{additional conditions:} \quad \bar{M}\delta w|_{a_2}^{b_2} - M\bar{w}|_{a_2}^{b_2} = 0.
\]
\[ M = EI \kappa; \]
\[ \kappa = -\frac{d^2 w}{dx^2} \in a, b; \]
\[ w, w^{(b)}_{a_2} = \bar{w}, \bar{w}^{(b)}_{a_2} \]
are the equations of strain compatibility and kinematic boundary conditions.

After appropriate transformations the variational problems of Lagrange and Castigliano will look like

\[ \delta \Pi^L (w) = \delta_w \Pi_1 = \]
\[ = -\int_{a}^{b} \left( \frac{d^2 M}{dx^2} + q \right) \delta w dx + \]
\[ + (M' - \bar{M}') \delta w_{a_2}^{(b)} - (M - \bar{M}) \delta w_{a_2}^{(b)} = 0 \]
and includes equilibrium equation (Euler equation) and natural (static) boundary conditions.

\[ \delta \Pi^C (M) = \delta_M \Pi_2 = \]
\[ = -\int_{a}^{b} \left( \kappa + \frac{d^2 w}{dx^2} \right) \delta M dx + \]
\[ + (\bar{w} - w) \delta M_{a_2}^{(b)} - (\bar{w} - w') \delta M_{a_2}^{(b)} = 0 \]
and includes strain compatibility equation (Euler equation) and natural (kinematic) boundary conditions.

Formulate the Lagrange and Castigliano principles as follows.

**Lagrange principle**

Of all the possible systems of displacements real displacements provide the Lagrange functional with stationary (minimum) value. Under these conditions possible displacements are meant as those which satisfy the strain compatibility equation and the equation of constraints (kinematic boundary conditions).

**Castigliano principle**

Of all the possible forces the real forces provide the Castigliano functional with stationary (maximum) value. Under these conditions the possible forces are meant as those which satisfy the equilibrium equation and static boundary conditions.
As to the history of the issue, note that the term Lagrange principle is common in Russian and Soviet literature. In *Analytical Mechanics* (1788) J.-L. Lagrange obtains all forms of equations of absolutely rigid body under the effect of forces from the general formula of statics. In his theorem, which J.-L. Lagrange called *equilibrium properties that relate to the maximum and minimum*, he considered the case where the left side of his general formula of statics is a complete differential of a certain function $\Pi$ that depends on the coordinate system (in the case of solids). In modern terminology, this function $\Pi$ is the potential energy of the system. Equilibrium conditions of the system, described by this function are reduced to zero equality of its first complete differential. So for such systems the equilibrium states coincide with the positions in which the function $\Pi$ has a minimum or maximum. J.-L. Lagrange shows that the equilibrium state that meets the minimum of function $\Pi$ is stable and equilibrium state, which corresponds to maximum of the function $\Pi$, is unstable. He does not consider the case $\Pi = \text{const}$ and the case of minimax. J.-L. Lagrange has formulated from the variational equation – the principles of possible velocities (displacements) – the corresponding variational principle for some potential function, which much later was also written for deformable bodies and was called the full potential energy of the system. We know that the total potential energy of elastically deformed body equals the work of elasticity forces in the transition of the given system state to the state with zero deformation.

J.-L. Lagrange’s proof contained some defects; Peter Gustav Lejeune Dirichlet (1805 1859) removed them and thus formulated the well-known Lagrange-Dirichlet principle.

Recall that the term «vis viva» (manpower), was first used by G.V. Leibniz. The term energy was introduced by Thomas Young in 1807 (in *A Treatise on Natural Philosophy*, Lecture VIII) and the term work by G.-G. Coriolis. W.R Hamilton wrote in his letter to P.G.Tait in 1862: "The Energy and Work in their old English sense – these are things I know. But I have only the most vague idea of the
The current meaning of the terms" [Graves, 1882-1889, Vol. III, p. 150].

In the 19th century Scottish engineer and physicist William John Macquorn Rankine (1820-1872) introduced the concept of potential energy and the German engineer Friedrich Engesser – the term complementary potential energy [Timoshenko, 1957].

The concept of mechanical work arose in close connection with the study of machines. "I am pleased with this important example of fruitful effects of a purely technical problem – in this scenario the question of the beneficial effect of machines – on theoretical research," wrote F. Klein [Klein, 1937, p. 109].

In the west the formulas for $\Pi^L(u)$ and the corresponding variational principle are more often called Dirichlet and Green variational theorem, since they are connected with the concept of elastic potential introduced to the theory of elasticity by George Green (1793-1841) [Green, 1739] and the principle of elastic equilibrium proved by P.G.L. Dirichlet [Dirichlet, 1846]. According to P.G.L. Dirichlet if applied forces are potential, the elastic equilibrium is stable if and only if the total potential energy of the mechanical system $\Pi$ is minimal, for example, the center of gravity is at the lowest point (Torricelli principle). This is the principle of minimum of potential energy. The Dirichlet theorem of stability for the system of $n$ solids was proved by George Hamel (1877-1954) [Hamel, 1912, p. 485-487].

For example, if the central tension-compression the functional of the full potential energy of the system is as follows [Bazhenov, 2014, p. 229]:

$$\Pi^L(x,u,u') = \frac{1}{2} \int_a^b E I u'^2 dx - \int_a^b q_x u dx \rightarrow \text{min},$$

$$\delta \Pi^L(u) = \frac{\partial \Pi^L}{\partial u'} \delta u' + \frac{\partial \Pi^L}{\partial u} \delta u = 0.$$  

P.G.L. Dirichlet has analyzed energy criterion of stable equilibrium for the systems with the infinite number of degrees of freedom.
\[ \delta^2 \Pi_{DG} \begin{cases} < & \rightarrow \text{unstable} \\ = & \rightarrow \text{neutral} \\ > & \rightarrow \text{stable} \end{cases} \rightarrow \text{equilibrium} \]

In this energy criterion \( \delta^2 \Pi_{DG} \) – the second variation of the full potential energy is to be variated twice.

Generally, in mathematical physics the Dirichlet principle is referred to potential theory and formulated as follows: if, for example, the function \( u(x) \) is the solution of Poisson's equation

\[ \Delta u + f = 0 \]

in the area \( \Omega \in R^n \) with the boundary condition \( u = g \) at the boundary \( \partial \Omega \), then \( u(x) \) can be found as the solution of the variational problem on a minimum

\[ [v(x)] = \int_\Omega \left( \frac{1}{2} |\nabla v|^2 - vf \right) dx \]

among all twice differentated functions there are \( v \) such that \( v = g \) at the boundary \( \partial \Omega \).

This statement was formulated (but not proven) by P.G.L. Dirichlet. Karl Weierstrass showed that in some cases the Dirichlet principle is wrong; then the conditions of its use were specified by B. Riemann, A. Poincare, D. Hilbert and others [Berdychevsky, 1983], [Michlin, 1950], [Petrova, 1966].

The above examples show that the Dirichlet and Green variational theorem is the interpreted formalized theory in terms of calculus of variations. Formalism of variational calculus was used by the Göttingen school, headed by Felix Klein, to justify the elasticity theory. Klein managed to invite to Göttingen such famous mathematicians as David Hilbert in 1895, Hermann Minkowski (1864-1909) in 1902, Carl Runge (1856-1927), in 1904, Edmund Landau (1877-1938) in 1909, D. Hilbert in 1899 proved the existence of the solution of Dirichlet variational theorem [Hilbert, 1901]. S.P. Timoshenko came to Göttingen in spring 1909, during all the

The beginning of a broad energy trend in solving the elasticity problems served the publication of Clapeyron theorem about the actual work of elastic forces (1852). Clapeyron formula not only allows for a gradual increase of internal forces in the process of the body deformation, but also gives the work dependence on stress for any elastic body.

The next stage in the theory of elastic bodies consisted in the discovery of works reciprocity (A.L. Cauchy, 1857, J.K. Maxwell, 1864; E. Betty, 1872).

The term Castigliano principle in Western literature is also more often called variational theorem of Menabrea and Castigliano.

Leaving aside the historical priorities, the above terms for one-, two- and three-dimensional problems can be called the principles of minimum of potential energy and minimum (with allowance for a sign) of complementary potential energy, that emphasizes their duality [Vasydzu, 1987]. If appropriate formulations present definitions of possible displacements and forces (or stresses), the minimum can be defined as the absolute one.

This formulation of the Catigliano principle is hereinafter related to the preservation of the mathematical methodology of Legendre, Young-Fennel-Lagrange, following which the Castigliano variational problem is considered as a dual problem to direct variational problem of Lagrange and vice versa. As already noted, the extreme values of Lagrange and Castigliano functionals coincide, when the former reaches a minimum, and the other – maximum. If the second dual function (functional) is chosen in a form \( \Pi^K(M) = \int_a^b \frac{M^2}{2EI} dx \), it, accordingly, reaches a minimum, and that is the principle of least work known in mechanics.
By R. Courant, D. Gilbert [Courant, Gilbert, 1951b] “the importance of Castigliano principle is that theoretically it is a very important example that confirms a general law of duality of variational problems.”

Luigi Donati was the first, who noticed that $\Pi^L(u)$ and $\Pi^C(\sigma)$ in the Lagrange and Castigliano formulas were functionals in certain formulations. In his works [Donati, 1888, 1889, 1894] he clearly explained the relationship between functionals and concepts of elastic potential in terms of the elasticity theory and and calculus of variations.

The requirement of identical equality to zero of the left side with allowance for the principle of independence of variations gives in a form of natural conditions the equilibrium equation as the Euler equation

$$\frac{d^2 M}{dx^2} + q = 0$$

and natural static boundary conditions, which are insufficient in the set additional conditions.

Lagrange variational equation

$$-M_b'\delta w_b - M'_a \delta w_a - M_b \delta w_b + M_a \delta w_a +$$

$$+ \int_{a}^{b} \left( \frac{d^2 M}{dx^2} - q \right) \delta w \, dx = 0,$$

considering that the resultant $R$ is under the integral

$$R \, dx + q \, dx + dQ = 0 \quad \Rightarrow \quad R = -\frac{dQ}{dx} - q = -\frac{d^2 M}{dx^2} - q,$$

is a principle of possible displacements, namely, if the sum of works of all the forces acting on the system at any possible displacements is zero, the system is in equilibrium. This refers for possible
displacements described by smooth continuous functions and satisfy the conditions of constraints\textsuperscript{29}.

**Castigliano variational equation**

\[ w_b \delta M'_a - w_a \delta M'_a - w'_b \delta M_b + w'_a \delta M_a + \int_a^b \left( \kappa + \frac{d^2 w}{dx^2} \right) \delta M dx = 0 \]

is the principle of possible forces, if the sum of works carried out at any possible changes of forces is zero, the system satisfies the strain compatibility equation. Under these conditions statically possible systems of forces are considered as possible.

In the case of inhomogeneous boundary conditions

\[ w, w'|_{a_2} = \bar{w}, \bar{w}'|_{a_2} \ 	ext{and} \ M, M'|_{a_1} = \bar{M}, \bar{M}'|_{a_1} \]

We obtain, respectively:

**Lagrange variational equation**

\[
(M' - \bar{M})\delta w|_{a_1} - (M - \bar{M})\delta w'|_{a_1} + \int_{a_1}^{b_1} \left( -\frac{d^2 M}{dx^2} - q \right) \delta w dx = 0.
\]

**Castigliano variational equation**

\[
(w - \bar{w})\delta M|_{a_2} - (w' - \bar{w}')\delta M'|_{a_2} + \int_{a_1}^{b_1} \left( \kappa + \frac{d^2 w}{dx^2} \right) \delta M dx = 0.
\]

Thus:

We have additional conditions (restrictions) in Lagrange principle

We have additional conditions (restrictions) in Castigliano principle

---

\textsuperscript{29} Some books in structural mechanics, when formulating the beginning of possible displacements, include additional requirement of the infinite smallness of those displacements on which we calculate possible work. That is essential only for geometrically nonlinear posing of the problem.

In a number of literary sources the term “possible work” is used instead of the word “work”. In so doing they mean in the expounding sense the abstraction, which differs from real work, since forces conducting work may belong to one state of the system, while corresponding displacements – to other one. At the same time this notion is defined in the section devoted to the principle of possible displacements, as the works of forces in possible displacement, though formulation of the above principle does not include the term “possible work”, and the word “work” is used instead of it. Similarly, possible work is also defined in the classical course by Appel [Appel, 1960]. The term possible work is not used as a rule in formulation of the principle of possible displacements in other sources as well [Lanczos, 1965], [Lurie, 1970], [Novozhilov, 1958], etc.
\[ M = EI\kappa; \quad \kappa = -\frac{d^2w}{dx^2}; \]

\[ w, w|_{a_2}^{b_2} = \tilde{w}, \tilde{w}|_{a_2}^{b_2} \]

and obtain natural conditions

\[ \frac{d^2M}{dx^2} = -q; \quad M, M|_{a_1}^{b_1} = \tilde{M}, \tilde{M}|_{a_1}^{b_1}. \]

That is the additional conditions of one variational problem are natural for the other one and vice versa. Such problems form a pair of dual variational problems. They are solved either by solution of each variation problem alone or by joint solution of equations of natural (additional) conditions of the both problems. The presented dependences are explained by formal conjugacy of operators of differential equilibrium equations \( \frac{d^2M}{dx^2} = -q \) and strain compatibility

\[ \frac{d^2w}{dx^2} = -\frac{M}{EI} \]

that appears in the Green formula

\[
\int_a^b \frac{d^2M}{dx^2}wdx = Mw|_a^b - Mw|_a^b + \int_a^b M \frac{d^2w}{dx^2}dx,
\]

and as to the physical meaning is the result of Clapeyron theorem and the law of conservation of energy.

Using the Lagrange multiplier method we can interchange additinal and natural conditions, i.e. obtain Castigliano functiona from the Lagrange functional and vice versa. This transformation in the calculus of variations is called Friedrichs transformation. The theory of functional transformations is described in [Courant, Gilbert, 1951b]. Note that extreme values of the Lagrange and Castigliano functionals and of all functionals, obtained using Lagrange multipliers coincide.

Lagrange functional \( \Pi = U + A \) is called full potential energy of the system, which is the sum of potential energy of elastic
deformation and work of external forces. This elastic deformation potential energy $U$ is calculated as the work of internal forces is considered positive, and the work of external forces $A$ is calculated as the product of force by displacement and is considered negative. Sometimes the full potential energy of the system is identified with work of the external and internal forces in the system transition from the strain state to the original one. Note that the functional (or function) $\Pi$ can be obtained as a dual one by Young to complementary potential energy, i.e. work of the internal forces.

The theory of R. Courant and D. Hilbert [Courant, Hilbert, 1951a] is based on the following provisions. Any of the conditions of functional stationarity may be included in additional conditions; such a variational problem will be equivalent to the original one. The second provision is to use the Lagrange multiplier method to take account of additional conditions and to obtain equivalent variational problems. In numerous problems, for example, for convex functions, the use of these provisions allows us to track also the change of extreme properties of functionals. In a number of problems without restrictions one can artificially introduce additional conditions for further introducing them in the functional using Lagrangian multipliers and conducting further transformations. This enables obtaining different formulations of the same variational problem with different variables and, in particular, making the important Friedrichs transformation.

The R. Courant and D. Hilbert theory of transformation of variational problems allows one to bring different functionals with additional conditions into agreement with each other and to derive a complete functional without any additional conditions from which one can obtain all possible functionals with additional conditions, as particular cases, and can formulate relevant partial variational principles.

Complete functionals are the functionals for which a variational problem is formulated without any additional conditions and covers all the components of the state space. Thus, under these conditions
the basic state space means a series of fields of displacements, strains, stresses (forces).

The complete functional is the most general energy characteristic of the system, since, on the one hand, all possible partial functionals in the given space can be obtained of the complete functional, on the other hand – this functional is enough for determining all the components of the fields of displacements, strains, stresses (forces), that is, for complete solving the problem in a given space of states.

The functionals, for which the variational problem is formulated with additional conditions that determine subspace in the chosen state space, are called partial functionals.

Partial functionals are obtained from complete ones by introducing additional conditions for some components of the given state space.

Thus, in the chosen state space the concept of complete and partial functionals are determined and are of absolute character. However, in the transition from one space to another these concepts become relative. That is the complete functional, determined in some space, can be considered as partial in the expanded space. It is partial in respect of the complete functional in the extended space.

The general variational principle is formulated as follows: the actual fields of parameters of the stress-strain state of the system provide the complete functional with a stationary value.

*General variational theorem.* Euler equations and the natural boundary conditions (natural conditions of a functional) include a complete system of equations and boundary conditions of the given theory that are expressed through components of the corresponding space of states.

As a rule, you can give a more detailed formulation of the general variational principle: not just a stationary value but a minimax (or maximin, or saddle point) of the complete functional corresponds to the true stress-strain state of the system. The exceptions are functionals with neither extremes, nor minimaxes or maximins.

A partial variational principle is formulated as follows: the actual fields of parameters of the stress-strain state of the system that satisfy
the given restrictions in the form of additional conditions, provide the partial functional with a stationary value in these additional conditions, i.e. in the subspace of the given space of states. Typically, the partial functional has not just a stationary value, but has a conditional extremum or minimax, or maksimin, or saddle point.

*Partial variational theorem.* Euler equations and the natural boundary conditions of the problem by conventional stationary value of a partial functional (functional natural conditions) with additional conditions constitute a complete system of equations and boundary conditions of the theory.

### 6.3. Clapeyron theorem. Theorems that bind the volume and surface integrals. Integral formula.

**Papkovich formula**

The Clapeyron theorem can be written as follows [Bazhenov, 2014]

$$2U = \iiint_V \left[ \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{yz} \gamma_{xy} + \tau_{xz} \gamma_{yx} + \tau_{xy} \gamma_{yx} \right] dxdydz.$$

So we obtain

$$\iiint_V (Xu + Yv + Zw) dxdydz + \iiint_S (P_{xy} u + P_{yx} v + P_{zx} w) dS = 2U,$$

or in the matrix form

$$\iiint_V \sigma^T \varepsilon dV = \iiint_V g^T u dV + \iiint_{S_1} P_{S}^T u dS + \iiint_{S_2} u^T P_S dS.$$ 

That is for the real state of linearly elastic system, the equation of equilibrium, strain compatibility, physical aspect of the problem and boundary conditions being satisfied in this state, the double potential energy of elastic deformation is equal to the work of external forces. This provision is the Clapeyron theorem.

Given that, $\varepsilon = A^T u$, and $g^T = -(A\sigma)^T$ the expression for Clapeyron theorem

$$\iiint_V \sigma^T A^T u dV = -\iiint_V (A\sigma)^T u dV + \iiint_{S_1} P_{S}^T u dS + \iiint_{S_2} u^T P_S dS$$

341
is a theorem about divergence and in the above form allows you to transfer the operation of differentiation vector \( \mathbf{u} = \{u, v, w\}^T \) to vector

\[
\mathbf{g} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}^T
\]

and vice versa, that is often used for various productions variational formulations of the problems of elasticity theory [Rozin, 1998].

Equality of works of internal and external forces in the matrix form is expressed as follows

\[
\iint_V \int_V \mathbf{g}^T \mathbf{u} dV = \iint_{S_1} \int_{S_1} \mathbf{u} \mathbf{P}_s dS + \iint_{S_2} \int_{S_2} \mathbf{u}_s \mathbf{P}_d S , \quad (6.9)
\]

which is essentially a theorem about divergence.

Generally, the theorems that bind the volume and surface integrals are often used in structural mechanics and theory of elasticity. The main ones are listed below.

- **Theorem of divergence**
  \[
  \int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot dS.
  \]

- **Theorem of curl**
  \[
  \int_V \nabla \times \mathbf{F} dV = \int_S \mathbf{F} \times dS.
  \]

- **Theorem of gradient**
  \[
  \int_V \nabla \Phi dV = \int_S \Phi dS.
  \]

- **Green theorem**
  \[
  \int_F \nabla \Phi \cdot \nabla \Psi dV + \int_S \Psi \nabla^2 \Phi dV = \int_S \Psi \nabla \Phi \cdot dS ,
  \]
  \[
  \int_S (\Psi \nabla^2 \Phi - \Phi \nabla^2 \Psi) dS = \int_F \Psi \nabla \Phi \cdot (\Phi \nabla \Psi) - \Phi \nabla \Psi \cdot dS.
  \]

- **Particular cases**
  \[
  \int_V \nabla^2 \Phi dV = \int_S \Phi \nabla \Phi \cdot dS = \int_S \Phi \frac{\partial^2 \Phi}{\partial n} dS \ (\text{Gauss theorem}),
  \]
  \[
  \int_V \left| \nabla \Phi \right|^2 dV + \int_S \Phi \nabla^2 \Phi dV = \int_S \Phi \nabla \Phi \cdot dS - \int_S \Phi \frac{\partial \Phi}{\partial n} dS.
  \]
To obtain the expression (6.9) one can use the following general integral value, which is essentially the Clapeyron theorem:

\[
\iiint_{V} a^T (Ab) dV = \iint_{S} a^T (A_{S} b) dS - \iiint_{V} (A^T a) b dV ,
\]

where \( a(x,y,z) = \{a_1, a_2, a_3\}^T \) – an arbitrary vector with three components that are functions of coordinates; \( b(x,y,z) = \{b_1, b_2, b_3, b_4, b_5, b_6\}^T \) – arbitrary vector with six components that also are functions of coordinates.

When deriving numerous general propositions of structural mechanics and elasticity theory, the formula obtained by P.F. Papkovych and bearing his name appears useful; the formula which is based on the expression for external forces is as follows:

\[
A_{3C} = \iiint_{V} (A\sigma + g)^T u dV + \iint_{V} (A^T u - \varepsilon)^T \sigma dV + \iint_{S} (P_S - A_S \sigma)^T u_S dS + \iiint_{V} \sigma^T \varepsilon dV .
\]

Note that the elements of four arbitrary states are used in formula (6.11), and the work of external forces, which corresponds to the first freely chosen stress state and performs the work on displacement of the second arbitrary stress state, is combined, somewhat artificially, with stress components of the third and strain components of the fourth arbitrary stress state.

P.F. Papkovych has noted that the obtained relationship, which seems at first glance rather artificial, is actually very convenient both to prove a series of general theorems of elasticity theory, and to estimate these theorems in a number of other basic dependences of the elasticity theory. All those general theorems of the elasticity theory, which were to be considered, can be derived directly from the equality (6.11). To do this the four stress states, presented in (6.11) as arbitrary stress states, should be brought each time in certain correlation with each other.
Schemes of obtaining partial functionals from the full one\textsuperscript{30}:

\[
\Pi(\mu, \varepsilon, \sigma)
\]

Stationary conditions: geometric, static and physical equations

\[
\Pi(\mu)
\]
\[
\Pi(\mu, \varepsilon)
\]
\[
\Pi(\varepsilon, \sigma)
\]
\[
\Pi(\varepsilon, \sigma)
\]
\[
\Pi(\sigma)
\]

Additional conditions

geometric physical
geometric static
static physical

Use of two groups of geometric, static and physical equations as additional conditions for obtaining partial functionals from a full one

While variation of displacements is performed in the functional of the Lagrange (Dirichlet and Green) variational theorem, the internal forces or stresses belong to varying ones in variational theorem of Castigliano (Menabrea and Castigliano). In other words, the variational theorem of Menabrea and Castigliano is a functional of internal forces or stresses. It is known that the substantiation of the classical structural theory was accompanied by a dispute, which concerned mainly general practical suitability of different forms of the Menabrea and Castigliano variational theorem, such as the Maxwell-Betti reciprocity equation, principle of possible changes in the stress state or the second Castigliano theorem. E. Hellinger and G. Prange (1885-1941) took part in this debate among others.

\textsuperscript{30} Polak’s expression comes to mind "... science has its own aesthetics and beauty of the logical symmetry of variational principles of mechanics, which cannot help but capture mathematicians, physicists, engineers." (From the preface to the book: K. Lanczos \textit{Variational Principles of Mechanics}).

344
In his review of *Die allgemeinen Ansätze der Mechanik der Kontinua* (The general approach to continuum mechanics), released in 1914 in the *Encyclopedia of Mathematical Sciences*, edited by Klein and K.H. Müller, E. Hellinger has shown for the case of three-dimensional continuum, how the principle of minimum potential energy can first be transformed to its canonical form using the canonical transformation of analytical mechanics, while displacement and stress variables become unknown variables of the state. Then, E. Hellinger obtains the Menabrea and Castigliano variational theorem using equilibrium conditions as additional ones.

![Portraits of prominent physicists](image)

G. Prange made the next step in his dissertation, completed in 1915 in Göttingen, and in his qualification work *Extremum of the Deformation Work*, made in Hanover a year later, but, unfortunately, published only in 1999 [Kurrer, 2008]. In his dissertation G. Prange gives a mathematical justification of the theory of elasticity with the help of calculus of variations using Hamilton-Jacobi canonical transformation known from analytical mechanics. Both variable displacement $u$, and variable force or stress $\sigma$ are the unknown varied variables of the state (Fig. 6.2) in a new canonical variational problem, which appeared after the canonical transformation.

E. Reissner, who worked independent of E. Hellinger and G. Prange, published in 1950 his famous work on six pages *On the Variational Theorem of Elasticity* [Reissner, 1950]. In this article he developed, although not referring to the Hamilton-Jacobi theory, the
variational theorem, which is similar to the Hellinger and Prange theorem. M.E. Gurtin gave this variational theorem the name of Hellinger, Prange and Reissner and marked its by symbol $\Pi_{H,P,R}$. In the case of spatial $\Pi_{H,P,R}$ — that is a functional of three displacement vector components $u$ and six components of the stress tensor $\sigma$.

![Diagram](image)

**Fig. 6.2**

Remembering those days, E. Reissner wrote: "Having used the variational theorem for stresses and the variational theorem for displacements, I began to wonder whether this had to be an either-or proposition. The first consequence was a generalization of the variational theorem for stresses, to make this theorem applicable to linear problems of simple harmonic motion. The possibility of this generalization depended on the simultaneous introduction of stress and displacement variations, which had to be interdependent to retain the dynamic constraint stipulations. With the concept of independent stress and displacement variations, a natural next step was to think about the possibility of a variational theorem with independent stress and displacement variations".

In 1953 E. Reissner summarized his variational theorem on elastic continua with large displacements.
Solid mechanics describes the behavior of elastic continua using:
- stress tensor $\sigma$,
- displacement vector $u$,
- strain tensor $\varepsilon$.

Of these three state variables the stress tensor $\sigma$ and displacement vector $u$ appearing in such variational theorems:

- Menabrea and Castigliano $\Pi_{MC} (\sigma)$,
- Lagrange, Dirichlet and Green $\Pi_{DG} (u)$,
- Hellinger, Pranhe and Reissner $\Pi_{HPR} (\sigma, u)$.

Such was the level of knowledge of variational theorems in 1950. The three scientists quickly raised the question, whether we can formulate a general variational theorem in which functional the strain state appeared along with displacement and stress states. The answer was found by B.M. Fraeis de Veubeke (Belgium), Hú Hāichāng (China) and Kyuichiro Washizu (Japan), who worked independent of each other.
E. Reissner remembers how K. Washizu visited him in the period of his work at Massachusetts Technological Institute between 1953 and 1955 years and explained his variational theorem: “... my friend Washizu ... came one day to my office to say that he had a variational theorem with independent variations not only of stresses and displacements, but also of strains, in such a way that not only equilibrium and stress-strain, but also strain displacement relations came out as Euler equations. I first objected that since only stresses and displacements would be encountered in the boundary conditions of problems, it was not natural to consider strain displacement relations in ways other than as defining relations. I was, however, soon persuaded that the ‘three-field’ theorem which Washizu, and independently Hu, had proposed was a valuable advance which I wished I had thought of myself”. Therefore, this general variational theorem is called in the literature in honor of Hu Hâichâng and Washizu; it will be signed as $\Pi_{H,W} (\sigma, u, \varepsilon)$.

B.N. Fraeijs de Veubeke developed a variational theorem with four variable fields or variables states even in 1951. He called this theorem general variational principle. In his theorem not only stress, displacement and strain vary independent of each other, but also surface load $t$ ($t_i$ in index notation). In honor of B.M. Fraeijs de Veubeke, K. Washizu and
Hú Hâichâng this functional is referred to as $\Pi_{F,H,w}(\sigma,u,\varepsilon,t)$; 
\[ t_i = \sigma_{ij} \cdot n_j \] (\(n_j\) – vector of normal to the surface, \(\sigma_{ij}\) – stress tensor).

In Fig. 6.3, \(a\) the variation process following general variational theorem by B.N. Fraeijs de Veubeke, in Fig. 6.3, \(b\) – the process of variation following variational theorem by Hellinger, Pranhe and Reissner, and in Fig. 6.3, \(c\) one more variational theorem presented by B.N. Fraeijs de Veubeke where stress \(\sigma\) and strain \(\varepsilon\) vary.

![Diagram](image)

Fig. 6.3

Synopsis of lectures by Klaus Knothe on finite element method in design calculations, which he read in the Aerospace Department of the Berlin Technical University, contains a systematic and very attractive presentation of seven variational theorems. This synopsis also contains a diagram which brilliantly illustrates the correlation between seven variational theorems except $\Pi_{F,H,w}(\sigma,u,\varepsilon,t)$ (Fig. 6.4).

On this diagram

- vertical hatching corresponds to variation of a field variable or state \(\varepsilon\) (deformation): $\Pi(\varepsilon)$;
- Diagonal hatching to the upper left downwards to the right represents a variable of a field or state $u$ (displacement), that is the variational theorem of Lagrange, Dirichlet and Green: $\Pi_{D,G}(u) = \Pi(u)$;

- diagonal hatching to the upper left upwards to the right represents a variable of the field or state $\sigma$ (stress), that is the variational theorem of Menabrea and Castigliano: $\Pi_{M,C}(\sigma) = \Pi(\sigma)$;

- hybrid variational theorems lie in the regions of the hatchings intersection: variational theorem of Hellinger, Pranhe and Reissner $\Pi_{H,P,R}(\sigma, u) = \Pi(\sigma, u)$ – where diagonal hatchings intersect, theorem of Fraeijs de Veubeke, Washizu and Hú Háichăng $\Pi_{F,H,H}(\sigma, u, \epsilon) = \Pi(\sigma, u, \epsilon)$ – where all types of shading intersect, additional variational theorem of Fraeijs de Veubeke $\Pi_{F}(\sigma, \epsilon) = \Pi(\sigma, \epsilon)$ – where intersect the lines that go to the upper left upwards.

Fig. 6.4

350
6.4. Conclusions

Various partial functionals, as involuntary variational problems with additional conditions, may be obtained from a complete functional. Any expressions of Euler equations and natural boundary conditions, realizing stationary value of complete functional (free variational problems) may be taken as additional conditions. Meeting additional conditions beforehand, i.e. before the variation, and eliminating with their help some part of functional arguments of the first functional, we obtain the corresponding partial functional.

Partial variational principles state that of these possible fields of stress and strain state of elastic bodies that meet additional conditions, there are really only those providing the appropriate functional with a stationary value.

For the variational equation with some additional conditions, the Euler equations are those equations and natural boundary conditions, which jointly with additional conditions constitute the entire scheme of equations and boundary conditions, i.e. the Euler equations and boundary conditions for the complete variational equations.
As an example we can present Lagrange, Castigliano, Reissner partial variational principles, boundary conditions and etc.

The Lagrange principle is obtained, if physical and geometrical equations as well as geometric boundary conditions are taken as additional conditions. That is those displacement functions are possible that meet the above additional conditions. And those possible displacements are true, which provide the Lagrange functional with a stationary value. Under these conditions the Euler equations are static, and there are also natural (static) boundary conditions.

Castigliao principle is obtained if physical and static equation as well as static boundary conditions are taken as additional conditions. That is those functions are possible which meet these additional conditions. And those possible stress (force) functions are valid that assign a stationary value the Castigliao functional, therewith the geometric equations and natural (geometric) boundary conditions are the Euler equations. The Reissner principle is obtained, if physical equations are taken as additional conditions. If all the static, physical and geometrical equations are taken as additional conditions, we get the functional boundary conditions. All boundary conditions proceed from this functional as stationarity conditions.

If, for example, we have to do with the spatial problem of elasticity theory, the general variational equations is written as follows [Bazhenov, 2014]:

\[
\iiint_{V} (\sigma - \mathbf{\varepsilon} \mathbf{u}_0) \delta \mathbf{\varepsilon} dV + \iiint_{V} (\mathbf{\varepsilon} - \frac{\partial \mathbf{u}_0}{\partial \mathbf{\varepsilon}}) \delta \sigma dV - \iiint_{V} (\mathbf{\varepsilon} - \mathbf{A}^T \mathbf{u}) \delta \sigma dV + \\
+ \iint_{\partial V} (\mathbf{A} \sigma + \mathbf{g}) \delta \mathbf{u} + \iint_{\partial S_1} (\mathbf{P} - \mathbf{P}_g) \delta \mathbf{u} dS + \iint_{\partial S_2} (\mathbf{u} - \mathbf{u}_g) \delta \mathbf{P} dS = 0.
\]

The general variational principle looks as follows:

\[
\delta \Pi^{\text{gen}}(\sigma, \varepsilon, \mathbf{u}) = 0.
\]

Additional conditions are absent.
Euler equations give equations of the boundary value problem of elasticity theory:

**In displacements**

\[ \sigma = D \varepsilon = D A^T u, \]
\[ A D A^T u + g = 0 \in V, \]
\[ A_S D A^T u = P_S \in S_1, \]
\[ u = u_S \in S_2. \]

Lagrange principle

\[ u_0 = \varepsilon^T D \varepsilon, \]
\[ \delta \Pi^L (u) = 0. \]

**In displacements and stresses**

\[ \{ A \sigma + g = 0 \}
\[ A^T u - D^{-1} \sigma = 0 \in V, \]
\[ A_S \sigma = P_S \in S_1, \]
\[ u = u_S \in S_2, \]

Castiglione principle

\[ u_0^{\text{comp}} = \sigma^T D^{-1} \sigma, \]
\[ \delta \Pi^C (\sigma) = 0. \]

**Variational equations**

\[ \iiint_V (A \sigma + g)^T \delta u dV + \]
\[ + \iint_{S_1} (P - P_S)^T \delta u dS = 0. \]
\[ - \iint_V (\varepsilon - A^T u)^T \delta \sigma dV + \]
\[ + \iint_{S_2} (u - u_S)^T \delta P dS = 0. \]

**Functionals**

\[ \Pi^L (u) = \frac{1}{2} \iiint_V \varepsilon^T D \varepsilon dV - \]
\[ - \iint_{S_1} g^T u dV - \iint_{S_2} P_S^T u dS. \]
\[ \Pi^C (\sigma) = -\frac{1}{2} \iiint_V \sigma^T D^{-1} \sigma dV + \]
\[ + \iint_{S_2} u_S^T P dS. \]

**Additional conditions**

\[ \sigma = D \varepsilon, \quad \varepsilon = A^T u \in V, S, \]
\[ u = u_S \in S_2. \]

Euler equations represent the equilibrium equation

\[ A \sigma + g = 0, \quad A_S \sigma = P_S, \]
which jointly with additional conditions: \[ \varepsilon = A^T u \in V, S, \]
\[ \sigma = D \varepsilon \in V. \]

Additional conditions

\[ \sigma = D \varepsilon, \quad A \sigma + g = 0 \in V, \]
\[ A_S \sigma = P_S \in S_1. \]

Euler equations are the equation of strain compatibility

\[ \varepsilon - A^T u = 0, \quad u = u_S, \]
which jointly with additional conditions: \[ A \sigma + g = 0 \in V, \]
\[ \sigma = D \varepsilon \in V. \]

353
provide a complete system of boundary value problem equations of elasticity theory.

Of the general variational equations we can obtain forms I and II of Hu-Washizu functional and corresponding variational equations

**Form I of Hu-Washizu functional**

\[
\Pi_1^{HW}(\sigma, \varepsilon, u) = \frac{1}{2} \iiint_V \epsilon^T D\epsilon dV + \\
+ \iiint_V \sigma^T (A^T u - \varepsilon) dV - \\
- \iiint_V g^T u dV - \iiint_{S_1} P^T S u dS - \\
- \iiint_{S_2} (u - u_S)^T A_S \sigma dS.
\]

\[
\delta \Pi_1(\sigma, \varepsilon, u) = 0.
\]

Additional conditions are absent.

Reissner variational equation

\[
- \iiint_V (\varepsilon - A^T u)^T \delta \sigma dV - \iiint_V (A\sigma + g)^T \delta u dV + \\
+ \iiint_{S_1} (P - P_S)^T \delta u dS + \iiint_{S_2} (u - u_S)^T \delta P dS = 0.
\]

Precondition

\[
\sigma = \frac{\partial u_0}{\partial \varepsilon} = D\varepsilon.
\]

Euler equations jointly with the precondition give a complete system of boundary value problem equations of the elasticity theory.

From Reissner’s variational equation we can obtain forms I and II of the Reissner functional and appropriate variational equations.
\[ u_0 = 2U - u_0^{\text{comp}} = \sigma^T A^T u - \frac{1}{2} \sigma^T D^{-1}\sigma, \]
\[ \Pi_1(\sigma, u) = \int_V \left[ \sigma^T A^T u - \frac{1}{2} \sigma^T D^{-1}\sigma \right] dV - \int_V g^T u dV - \int_{S_1} P_S^T u dS - \int_{S_2} (u - u_S)^T A_S \sigma dS. \]
\[ \delta \Pi_1^R (\sigma, u) = 0. \]
Additional condition \( \sigma = D\varepsilon. \)

Functional of boundary conditions

The general variational equation can be represented as:
\[ \delta \left( \frac{1}{2} \int_V \sigma^T A^T u d\chi \right) - \delta \left( \int_V u^T g dV \right) - \int_{S_1} P_S \delta u dS + \int_{S_2} (u - u_S) \delta P dS = 0. \]

Functional of boundary conditions (form I)
\[ \Pi_1^T (\sigma, u) = \frac{1}{2} \int_V \sigma^T A^T u dV - \int_V u^T g dV - \int_{S_1} P_S^T u dS + \int_{S_2} (u - u_S)^T A_S \sigma dS, \quad \delta \Pi_1^T (\sigma, u) = 0. \]
Additional condition:
\[ A\sigma + g = 0 \quad \in V, \quad \varepsilon = A^T u \quad \in V, \quad \sigma = D\varepsilon \quad \in V. \]

Variational equation:
\[ -\int_V \sigma^T \delta \varepsilon dV + \int_V u^T g dV + \int_{S_1} P_S \delta u dS = 0. \]

Functional:
\[ \Pi_2^T (\sigma, u) = -\frac{1}{2} \int_V \sigma^T A^T u dV + \int_{S_1} u^T (A_S \sigma - P_S) dS + \int_{S_2} u^T A_S \delta \sigma dS, \]
\[ 355 \]
\[ \delta \Pi_2^G (\sigma, u) = 0. \]

Additional conditions:

\[ A \sigma + g = 0 \quad \in V, \quad \varepsilon = A^T u \quad \in V, \quad \sigma = D \varepsilon \quad \in V. \]

The construction of a complete and partial functionals is also possible for elastic bodies fixed in some points, lines, planes, i.e. for multicontact systems.

Under these conditions a functional of boundary conditions for a multicontact problem is of interest. We can show that the classical methods of structural mechanics (methods of forces, displacements, a mixed method), the system of functionals for structural mechanics and various variants of the method of finite elements proceed from the functional of boundary conditions of multicontact problem. Indeed, let us divide the system (continual or rod) into the elements, which we will combine at some points. Take the fulfillment of the static, geometric and physical equations inside each element as additional conditions. If the problem is linear, then it is possible to construct stiffness or compliance matrices of finite elements. The problem solution is obtained by using a functional of boundary conditions, the boundary conditions of algebraic equations of classical structural mechanics proceeding from it as natural. If the shape, dimensions of the common elements and relationships between them are taken as those that take place in the classical approaches of structural mechanics, these approaches will not differ.

In structural mechanics the expressions for potential energy of elastic deformation and complementary potential energy are positively defined by quadratic forms. They are dual according to Young and related by Legendre transformation, and their corresponding values coincide [Bazhenov et al., 2013]. Legendre transformation is a particular case of the Young inequality and in this case, as to the physical meaning, is the equality of work of internal and external forces and corresponds to the Clapeyron theorem. Formulations of direct and dual, in the Legendre, Young-Fennel, Lagrange sense, variational problems are realized in the form of basic

356
variational principles – Lagrange and Castigliano variational principles, Lagrange and Castigliano theorems and they result in the systems of algebraic equations, which matrices (Hesse matrix) consist of second derivatives in correspondence with potential energy of elastic deformation (stiffness matrix) and complementary potential energy (compliance matrix). The above matrices are positive and meet certain Sylvester criteria, all their minors are positively determined [Belenky, 1964]. Moreover, matrices are mutually inverse. These considerations are also extended to functionals.

References
Aleksandrov, A.V., Lashchennikov, B.Ya. and Shaposhnikov, N.N. (1983), *Stroitelnaya mehanika. Nokostennyie prostranstvennyie sistemy* [Structural mechanics. Thin-walled space systems], Stroyizdat, Moscow, USSR.

Aleksandrov, A.V. and Potapov, V.D. (1990), *Osnyv teorii uprugosti I plastichnosti* [Bases of the theory of elasticity and plasticity], Moscow, USSR.

Appel, P (1960), *teoreticheskaya mehanika. T. 1* [Theoretical mechanics, V. 1], Transl. from 5th French edition by I.G. Malkin, Fizmatgiz, Moscow, USSR.


Bazarov, I.P. (2010), *Termodynamika* [Thermodynamics], Lan’, StPetersburg, Russia.

Belenkiy, I.M. (1964), Vvedenie v analiticheskuyu mehaniku [Introduction in theoretical mechanics], Moscow, Russia.

Berdichevsky, V.L. (1983), *Variatsionnyie printsipy mekaniki sploshnoi sredy* [Variational principles of continuum], Nauka, Moscow, USSR.

Bernshtein, S.A. (1957), *Ocherki po istorii stroitelnoi mehaniki* [Essays on the history of structural mechanics], Gosstroyizdat, Moscow, USSR.

Bernshtein, S.A. (1947), *Raschet konstruktivy s odnosteronnymi svazyami*
[design of structures with unilateral constraints], Stroyizdat, Moscow, USSR. Bekhterev, P. (1925), Analiticheskoye issledovaniye obobshchennogo zakona Guka. Primeneniyu ucheniya o potentialsnoi energii I nachala naimenshei raboty. Ch. I [Analytical study of generalizef Hook’s law. Application of the teaching on potential energy and principle of the least work. P. 1], Author’s publication.


Bubnov, I.G. (1912-1914), Stroitelnaya mekhanika korablya [Structural mechanics of the ship], Izdaniye morskogo ministerstva, St Petersburg, Russia.


Gvozdev, A.A. (1927), Obshchiy metod rascheta staticcheski neopredelimykh system. Teoriya I primery yevo primeneniya k raschetu ramnykh konstructsiy [General method of design of statically undeterminable systems. Theory and examples of its application to calculation of frame structures], MIIT, Moscow, USSR.


Duem, P. (1903), Razvitiye mehaniki [Development of mechanics.

Kirpichov, V.L. (1903), Lishnye neizvestnye v stroitelnoy mehanike [Redundant unknowns in structural mechanics], Izdatelstvo Kulzhenko, Kiev, Russia.

Kirpichov, V.L. (1917), Sobranie sochineniy. T. 1 [Collected works. Vol. 1], PPI, Petrograd, USSR.

Kirpichov, V.L. (1923), Soprotivlenie materialov. Uchenie o prchnosti postroek I mashin. Ch. 1, Ch. 2 [Material resistance. Study on strength of buildings and machines. P. 1, P. 2], Gosudarstvennoye izdatelstvo, Moscow, USSR.

Kirpichov, V.L. (1934), Lishnye neizvestnye v stroitelnoy mehanike [Redundant unknowns in structural mechanics], Gosteirizdat, Moscow-Leningrad, USSR.

Kirpichov, V.L. (1951), Besedy o mehanike [Discussions on mechanics], GITTL, Moscow-Leningrad, USSR.

Klein, F. (1937), Lektii o razvitiy matematiki v XIX stoletii [Lectures on development of mathematics in the 19th century], ONTI, Moscow-Leningrad, USSR.

**Navchalny posibnyk** [Settlement of invisible conflict in building structures. From intuitive and conceptual to scientific approach. Manual], Stal, Kyiv, Ukraine.


Kudrtavtsev, P.S. (1974), *Kurs istorii fiziki* [Course of the history of physics], Prosveshchene, Moscow, USSR.


Lanczos, C. (1965), *Variatsionnyie printsipy mekhaniki* [Variational principles of mechanics] Translated from English by V.F.Gantmakher, Mir, Moscow, USSR.

Leibzenon, L.S. (1943), *Variatsionnyie metody resheniya zadach teorii uprugosti* [Variational methods of elasticity theory], Gostekhizdat, Moscow, USSR.

Leibzenon, L.S. (1947), *Kurs teorii uprugosti* [Course of the theory of elasticity], Gostekhizdat, Moscow-Leningrad, USSR.

Leibniz, G.V. (1082), *Sochineniya v 4 t.* [Works in 4 vol], Moscow, USSR.

Lurje, A.I. (1970), *Teoriya uprugosti* [Theory of elasticity], Nauka, Moscow, USSR.

359

Mach, E. (1909), Mekhanika [Mechanics], St.Petersbourg, Russia.


Mikhlin, S.G. (1970), Variatsionnyie metody v matematicheskoi fizike [Variational methods in mathematical physics], Nauka, Moscow, USSR.

Moiseyev, N.D. (1961), Ocherki poistorii mekhaniki [Essays in the history of mechanics], Izdatelstvo AN SSSR, Moscow, USSR.

Nekrasov, N.V. (1907), K teorii ferm s zhestkimi soyedineniyami v uzlakh [On the theory of trusses with rigid joints in nodes], StPetersbourg, Russia.

Novozhilov, V.V. (1958), Teoriya uprugosti [Theory of elasticity], Sudpromgiz, Leningrad, USSR.

Papkovich, P.F. (1939), Teoriya uprugosti [Theory of elasticity], Oborongiz, Leningrad, USSR.

Perelmutter A.V. (2002), Zhili-byli [There was lived], Stal, Kiev, Ukraine; (2004), Stal, Kiev Ukraine.

Perelmutter A.V. and Slivker V.I. (2010), analiza [Design models of structures and possibility of their analysis], VVP Kompas, Kiev, Ukraine.


Perelmuter A.V. (2014), Besedy o stroitelnoi mekhanike [Discutpes on structural
mechanics], SCAD, SOFT, ACB, Moscow, Russia.
yestestvennykh nauk, MGU, Moscow, Iss. 5, pp. 200-218.
Pisarenko, G.S. (1991), Stepan Prokopovich Timoshenko, Nauka, Moscow,
Russia.
Plutarch. (1961), Sravnitelnye zhizneopisaniya. T. 1 [Comparative biographies.
V. 1], Izdatelstvo AN SSSR, Moscow, USSR.
Pogrebyssky, I.B. (1964), Ot Lagranzha k Einshteinu: Klassicheskaya
mekhanika XIX veka [From Lagrange to Einstein: Classical mechanics of the
19th century], Nauka, Moscow, USSR.
Pogrebyssky, I.B. (1966), Ot Lagranzha k Einshteinu [From Lagrange to
Einstein], Nauka, Moscow, USSR.
Pratusevich, Ya.A. (1948), Variansionnye metody v stroitelnoy mekhanike
[Variational methods in structural mechanics], OGIZ, Moscoq-Leningrad,
USSR.
Prigozhin, I. and Drefi, R. (1966), Khimicheskaya termodinamika [Chemical
thermodynamics], Nauka, Novosibirsk, USSR.
Rabinovich, I.M. (1950, 1954), Kurs stroitelnoi mekhaniki sterzhnevykh system.
T. 1 Staticheski opredelimye sistemy. T. 2 Staticheski neopredelimye
sistemy [Course of structural mechanics of rod systems. Vol.1 Statically
determinable systems. Vol. 2 Statically undeterminable systems],
Gosudarstvennoye izdatelstvoliteratury po stroitelstvu I arkhitekteure,
Moscow, USSR.
Nauka, Moscow, USSR.
Problemy mekhaniki sploshnoi sredy. K 70-letiyu acad. N.I. Mushelishvili
[On some variational theorems of the elasticity theory. Problems of
continuum mechanics On the 70 birthday of acad. N.I. Mushelishvili], AN
SSSR, Moscow, USSR.
Reitman, [M.I. (1973), Zalog prochnosti [Pledge of strength], Stroyizdat,
Moscow, USSR.
Rectoris, K. (1985), Variatsionnyie metody v matematicheskoi fizike I tekhnike
[Variational methods in mathematical physics and engineering], Mir ,
Moscow, USSR.
[Variational posing of problems for elastic systems], LGU, Leningrad,
USSR.
Rozin, L.A. (1986), Teoremy i metody statiki deformiruyemykh system [Theorems
and methods of statics of deformable systems], Leningrad, USSR.
Rozin, L.A. (1998), Zadachi teorii uprugosti I chislennie metody ikh resheniya [Problems of elasticity theory and numerical methods of their solution], SPbGTU, StPetersburg, Russia.

Rybakov, L.S. and Narchinsky, V.I. (1987), Variatsionnyie printsipy I metody stroitelnoi mekhaniki [Variational principles and methods of structural mechanics], Mai, Moscow, USSR.

Rayleigh, J. (1955), Teoriya zvuka T. 1, 2 [Sound theory, Vol. 1,2], Gostekhizdat, Moscow-Leningrad, USSR.


Slivker, V.I. (2005), Stroitelnaya mekhanika. Variatsionnyie osnovy [Structural mechanics. Variational principles], Idatelstvo ACB, Moscow, Russia.

Streletsky, N.S. (1921), K raschetu slozhnykh staticheski neopredelimykh system [On analysis of complex statically undeterminate systems], Moscow, USSR.


Tyulina, I.A. (1977), Joseph Lui Lagranzh [Joseph Louis Lagrange], Nauka, Moscow, USSR.

Tyulina, I.A. (1979), Istoriya I metodologiya mekhaniki [History and methodology of mechanics], Izdatelstvo MGU, Moscow, USSR.

Tyulina, I.A. and Rakcheyev, E.N. (1962), Istoriya mekhaniki [The history of mechanics], Izdatelstvo MGU, Moscow, USSR.

Umansky, A.A. (1935), Spetsialny kurs stroitelnoi mekhaniki. Ch.1 [Special course of structural mechanics. P. 1], USSR.

Wavell, U. (1867), Istoriya induktivnykh nauk ot drevneishego do nastoyashchego vremeni T. 1, 2, 3 [The history of inductive sciences from the very antiquity to the present. Vol. 1, 2, 3], StPetersburg, Russia.

Feodosyev, V.I. (1973), Izbrannye zadachi I voprosoy po soprotivleniyu materialov [Selected problems and questions on material resistance], Nauka, Moscow, USSR.

Feodosyev, V.I. (1973), Desyat lektsiy-besed po soprotivleniyu materialov [Ten lectures-disputes on material resistance], Nauka, Moscow, USSR.

Fepple, A. and Fepple, L. (1933, 1936), Sila i deformatsiya. T. 1, 2 [Force and deformation. Vol. 1, 2], Translated from German, ONTI, Moscow, USSR.


362
Filin, A.P. (1993), *Pyat chasov v obshchestve klassika nauki* [Five hours of intercourse with a classic of science], StPetersbourg, Russia.

Filin, A.P. (20070, *Ocherki ob uchonykh-mekhanikakh* Essays about scientists in mechanics], Izd. Dom Strategiya, Moscow, Russia.

Filonenko-Borodich, M.M. (1932), Osnovy teorii raboty uprugikh forces in plane systems, GTTI, Moscow, USSR.

Chekanov, A.A. (1982), Viktor Lvovich Kirpichov, Nauka, Moscow, USSR.

Cherepashinsky, M. (1888), *Kratkiy metodichsky ocherk razvitiya stroitelnoi mekhaniki* [ Brief historical essay of development od structural mechanics], Moscow, Russia.

Shekhter, R. (1971), Variatcionny metod v inzhernerykh raschotakh [Variational method in engineering design], Mir, Moscow, USSR.

Appell, P. (1900), Traité de Mécanique, III.


Boscovich, R.J. (1875), Philosophiae naturalis ad unicum legem virium in natura existentium, Venezia, Italy.


Castigliano, A. (1879), Theorie de l'équilibre des systemes elastiques, Turin, Italy.


Clapeyron, E. (1926), Abhandlung über die bewegende Kraft der Wärme, *Akademische Verlagsgesellschaft m.b.H.*


Crotti, F. (1877), Conversazioni – Saggi di critica scientifico-pratica, Minelli, Rovigo, Italy.


Donati, L. (1894), Ulteriore osservazioni intorno al theorema del Manabrea, Memorie dell’accademia discienze di Bologna, Tomo XV, ser. IV, p. 449.


Duhem, P. (1905), Les Origines de la statique, Paris, France.


Euler, L. (1744), Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, Marc Michel Bousquet.
Euler, L. (1767), De motu vibratorio tympanorum, Nove commentarii Academiae Scientiarum Imperialis Petropolitanae, Vol. 10.
Helmholtz, H. (1902), Dynamik continuierlich verbreiteten Massen, Verlag von Johann Ambrosius Barth, Leipzig, Germany.
Hertwig, A. (1933), Das "Kraftgroßenverfahren" und das "Formanderungsgroßverfahren" für die Berechnung statisch unbestimmter Gebilde, Der Stahlbau, Vol. 6, no. 19, pp. 145-149
Kurrer, K.-E. (2008), The history of the theory of structures, Ernst & Sohn Verlag für Architektur und technische Wissenschaften GmbH & Co. KG, Berlin, Germany.

Lamé, G. (1852), Leçons sur la théorie mathématique de l'élasticité des corps solides, Bachelier, Paris, France.

Lamé, G. (1859), Leçons sur les coordonnées curvilignes et leurs diverses applications, Mallet-Bachelier, France.

Mach, E. (1883), Die Mechanik in ihrer Entwicklung, historisch-critisch dargestellt, Leipzig, Germany.


Maxwell, J.C. (1927), The scientific papers, V. 1, P. 604, V. 2, P. 801.

Menabrea, L.F. (1842), Sketch of The Analytical Engine. Invented by Charles Babbage from the Bibliotheque Universelle de Genève, October, 1842, No. 82.

Menabrea, L.F. (1857), Principio generale per determinare le tensioni e le pressioni in un sistema elastico, Reale Academia delle Scienze di Torino.


Müller-Breslau, H. (1886), Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen, Baumgärtner’s Buchhandlung, Leipzig, Germany.


Navier, C.L.M.H. (1826), Résumé des Leçons données à l’Ecole Royale des Ponts et Chaussées sur l’Application de la Mécanique a l’Etablissement des Constructions et des Machines. 1st partie: Leçons sur la résistance des


Poisson, S.D. (1833), Traite de mecanique, V. 1-2, Paris, France.

Poncelet, J.V. (1826), Cours de mecanique appliquee aux machines, Paris, France.


Reissner, E. (1996), Selected works in applied mechanics and mathematics, Subdury: Jones & Burlett Publishers, Subdury, USA.


Schleusner, A. (1933), Das Prinzip der virtuellen Verruckungen und die Variationsprinzipien der Elastizitatstheorie, Der Stahlbau, Vol. 6, no. 19, pp. 145-149.
Washizu, K. (1975), Variational methods in elasticity and plasticity, Pergamon Press, USA.
Winkler, E. (1858), Formaenderung und Festigkeit gekrummter Koerper, insbesondere der Ringe, Der Civiligenieur, Bd 4, S. 232-246.
Winkler, E. (1862), Beitraege zur Theorie der Continentlichen Bruchentraeger, Der Civiligenieur, Bd. 8. S. 135-182.
Winkler, E. (1867), Die Lehre von der Elasticit"at und Festigkeit, H. Dominicus, Prague, Czechoslovakia.
Essay 7

DUAL NATURE OF THE PROBLEMS OF STRUCTURAL MECHANICS.
ON THE HISTORY OF THE FORCE METHOD AND DISPLACEMENT METHOD
The computer shapes the theory

J. Argyris
Introduction

Historically, the development of mechanics and, in particular, structural mechanics were related to geometry. For real objects were always preceded by geometric representations, schemes and constructions. Interestingly, that "... Archimedes himself – Plutarch wrote – thought the construction of machinery the occupation, deserving neither work nor attention; most of them came into being as if passing by, in the form of geometric entertainment and it is only because of King Hiero ambition, the latter persuaded Archimedes to distress not for long from theoretical considerations and refer to real things, to materialize somewhat his thought, to combine it with daily needs ... The well-known and favorite art of constructing mechanical instruments was initiated by Eudoxus and Archytas, who tried to resolve the issues, which proof only by considerations and drawings is too problematic; such is the problem of finding two middle proportionate quantities, to solve this problem the both used mechanical devices, when constructing the desired lines through arcs and segments. But since Plato was outraged, accusing them that they ruined dignity of geometry, mechanics was completely separated from geometry and, as far as it became one of the military sciences, philosophers took no interest in it for a long time" [Plutarch. 1961].

The dialectical relationship of mechanics and geometry confirms the famous saying of Isaac Newton: "Therefore Geometry is founded on mechanical practice and is nothing else than that part of universal Mechanics that proposes and demonstrates the art of measuring accurately. But since the manual arts are chiefly employed in the moving of bodies, it happens that Geometry commonly refers to their magnitude, and Mechanics to their motion. In this sense rational Mechanics will be the science of motions that result from any forces whatsoever, and also of the forces required to produce any motions, accurately proposed and demonstrated."
The connection between geometry and statics, by mechanics terminology – between the static and geometric sides of the problem, is clearly traced in history.

In his pioneer work *Analytical Mechanics*, J.-L. Lagrange has founded mechanics as a whole, and hence statics on a single principle: the principle of virtual velocities. J.-L. Lagrange has also recognized not only the equivalence of three principles of statics

- the principle of lever;
- the principle of virtual displacements;
- parallelogram of forces

but also made it clear that the principle of possible displacements can be mathematically transformed into a principle of equilibrium.

Statics, in the sense of body equilibrium at the level of theoretical mechanics is logically complete. This paved the way for the historical development of the theory of structures in the period of discipline formation.

The kernel of the theory, being the basis of kinematic concept of statics, which was founded by Aristotle and obtained its completed form owing to J.-L. Lagrange, is the principle of possible displacements. This principle has been successfully applied to simple mechanisms such as a lever, block or inclined plane. Leonardo da Vinci, for example, considered a stone arch as a mechanism. However, he did not calculate a thrust of an arch using the principle of possible displacements, but instead, proposed a method of its experimental determination.

Transformation of building structures into mechanisms for subsequent mechanical analysis was characteristic of the kinematic view of statics. The consequences of that mechanism is actually in equilibrium can be determined indirectly by analyzing the model in the deflected position, as is usually done when using the kinematic approach.

The kinematic concept of the statics [Kurrer, 2008] (Fig. 7.1, left), which was a component that combines Aristotelian theorem on the motion and natural philosophy, was rejected at the beginning of
anno Domini by Galileo and other scientists. Geometric approach to mechanics founded by Archimedes has become much more important. Here again we have to remember *The Dialogue* by Galileo. A cantilever beam, on which he demonstrated a bending fracture, became a metaphor for the geometric representation of statics.

<table>
<thead>
<tr>
<th>Kinematic presentation of statics</th>
<th>Vector presentation of statics</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Representatives:</em></td>
<td><em>Representatives:</em></td>
</tr>
<tr>
<td>Aristotle, Heron,</td>
<td>Archimedes, Heron,</td>
</tr>
<tr>
<td>Vitruvius,</td>
<td>Papp,</td>
</tr>
<tr>
<td>Tabit ibn Kurra,</td>
<td>Tabit ibn Kurra,</td>
</tr>
<tr>
<td>Nemorarii,</td>
<td>Guidobaldo del Monte,</td>
</tr>
<tr>
<td>Leonardo da Vinci,</td>
<td>Stevin, Galileo Galilei,</td>
</tr>
<tr>
<td>Tartalia, Cardano,</td>
<td>Roberval, Varignon,</td>
</tr>
<tr>
<td>Lagrange, Mohr,</td>
<td>Clapeyron,</td>
</tr>
<tr>
<td>Land, Müller-Breslau</td>
<td>Müller-Breslau</td>
</tr>
</tbody>
</table>

Fig. 7.1. Kinematic and geometric presentation of statics

While kinematic presentation of statics, as a purely theoretical idea in the sense of Plato, had high social prestige since old times, geometric approach to statics (Fig. 7.1, right) belonged to architecture and on this basis was considered as "the low-level art". Geometric representation of statics was developed based on Euclidean geometry and main practical requirements for the structures, where equilibrium was a natural condition that could not be eliminated without external damage. The fundamental concept of stability allowed the geometric presentation of statics to gain the dominant position in the theory of structures during the period of its formation (1825-1900). But such prominent civil engineers as A. Mohr, Robert Land (1857 1899), H. Müller-Breslau and others have made a significant contribution to cinematic presentation of statics, the development of which also was not interrupted. Differences between kinematic and geometric presentations of statics, discussions, which stimulated and accompanied the development of the theory of structures formed its most important elements.
The history of structural mechanics and the theory of structures are closely related to the concept of duality. At a stage of formation it mainly concerned the methods of graphical analysis based on projective geometry.

A funicular polygon and force polygon presented in Fig.7.2 are interchangeable, because no matter which of them is chosen as a force polygon, and which as a rope one, related to it. K. Culmann calls such figures the reciprocal ones. Working independently, J.C. Maxwell proved in 1864 that in case of non-concurrent forces such two figures are mutual, only when the force polygon can be considered as the polyhedron projection; another figure is therewith also the polyhedron projection. In Fig. 7.2 the two figures can be interpreted as projections of tetrahedral pyramids with their apexes at 0 and 0'. Knowledge of these mathematical relationships known also as duality of the funicular polygon and polygon of forces allowed Culmann to determine load functions only for arches of elliptical, parabolic and hyperbolic shape. Thus, a heuristic function based on projective geometry, remained the illustrative limited concept of scientific and technical theory of graphical statics [Kurrer, 2008].

It is well known that the conditions of equilibrium, law of material behavior and kinematic relationships give 15 partial
differential equations relative to 15 unknown scalar functions of three variables, namely:
- three displacements,
- six strains,
- six stresses.

The logical kernel of elasticity theory is characterized by this triune structure. Two approaches are used, when solving the elasticity theory problems: the exclusion of stresses and of displacements.

If in case of body linearity, homogeneity and isotropy deformations and stresses are excluded from the system of equations, the vector differential equation takes the form

$$\Delta \mathbf{u} + \frac{1}{(1 - 2 \cdot \nu)} \text{grad} (\text{div} \mathbf{u}) - \frac{2 \cdot (1 + \nu)}{E} \mathbf{g} = 0.$$  

This system of three partial differential equations relative to the displacement vector $\mathbf{u}$ at known volumetric forces $\mathbf{g}$ and two material constants $E$ (elasticity modulus) and $\nu$ (Poisson's ratio) plus geometric boundary conditions was named after H. Lame and L. Navier. By combining the equations and boundary conditions, we obtain a method for solving Lame-Navier differential displacement equations; the method is given a term the *displacement method* of mathematical theory of elasticity.

Another way is to eliminate displacements and deformations and to pass – again in the case of the body complete linearity, homogeneity and isotropy – to the tensor differential equation, named after E. Beltrami and J.H. Michell:

$$\Delta \sigma_{ij} + \frac{1}{(1 + \nu)} \frac{\partial^2 s}{\partial x_i \partial x_j} = -\left[ \text{grad} \mathbf{g} + \text{grad}^T \mathbf{g} + \frac{\nu}{(1 - \nu)} \cdot (\text{div} \mathbf{g}) \cdot \mathbf{I} \right].$$

Allowing for the force boundary conditions, the components of the stress tensor $\sigma_{ij}$ ($s$ – a sum of diagonal stress tensor components, $\mathbf{I}$ – identity tensor) can be determined from the system of six partial differential equations. The approaches leading to the solution of Beltrami-Michell differential stress equations, are called *the force method* of mathematical theory of elasticity.
In the literature, the first approach, using differential equations in Lame-Navier displacements and geometric boundary conditions (conditions that determine the displacement of the surface of the body), is called the first boundary problem and the second approach, using Beltrami-Michell differential stress equations and static boundary conditions (conditions that determine forces on the body surface) is called the second boundary value problem of elasticity theory. The third approach to solving problems of elasticity is also possible, when stresses are given in one part of the body surface \( s_1 \), and displacements – in the rest part \( s_2 \).

Eugenio Beltrami (1835-1900) – Italian mathematician known for his work in differential geometry and mathematical physics. Since 1871 he was engaged in research in the field of analytical functions and mechanics.

John Henry Michell (1863-1940) – Australian mathematician and engineer – had works in mathematics, physics, hydraulics and elasticity theory. He has established differential relationship between stress components in the elasticity theory (1899), solved the two-dimensional problem of the elasticity theory (1899).

7.1. The forms of expression of potential energy.

**Partial derivatives of potential energy**

Potential energy of a bar system loaded by forces \( P_1, P_2, \ldots, P_n \) is expressed by the formula [Rabinovich, 1954]

\[
U(P, \Delta) = \frac{1}{2} (P_1 \Delta_1 + P_2 \Delta_2 + \ldots + P_n \Delta_n),
\]

(7.1)

where \( \Delta_i \) indicated the total displacement of a force point \( P_i \) in the direction of the force.

In turn, the total displacements are expressed through unitary ones (i.e. through displacements in the same direction, caused separately by forces \( P_1=1, P_2=1, \ldots, P_n=1 \) so:

\[
\Delta_i = P_1 \delta_{i1} + P_2 \delta_{i2} + \ldots + P_n \delta_{in}.
\]

(7.2)
If we substitute the expressions $\Delta_i$ in formula (7.1), we get an expression for the complementary potential energy

$$U^{\text{comp}}(P) = \frac{1}{2}(\delta_{11}P_1^2 + \delta_{22}P_2^2 + \ldots + \delta_{nn}P_n^2) + (\delta_{12}P_1P_2 + \delta_{13}P_1P_3 + \ldots + \delta_{23}P_2P_3 + \ldots + \delta_{n-1n}P_{n-1}P_n)$$

or in short

$$U^{\text{comp}}(P) = \frac{1}{2} \sum \delta_{ii}P_i^2 + \sum \delta_{ik}P_iP_k, \quad (7.3)$$

in the second sum $i \neq k$. The right side of formula (7.3) is homogeneous algebraic polynomial of the second degree in respect of external forces. It includes squares of forces with some coefficients and their pairwise products. Such polynomials are called quadratic forms, so we can say that the potential energy of linearly deformable body can always be presented as a quadratic form of external forces.

The composition of external forces can also include reactions of such constraints, which rejection does not affect geometric invariability and immobility of the system.

Potential energy is always positive, hence the quadratic form (7.3) has such a property that at any values of variables $P_1, P_2, \ldots, P_n$ it cannot become negative. These quadratic forms are called positive definite.

Instead of considering external forces $P_1, P_2, \ldots, P_n$ as independent variables we can take total displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ as independent variables corresponding to these forces. These two
different possibilities are illustrated by a simple example (Fig. 7.3). In Fig. 7.3, the forces $P_1, P_2, \ldots, P_n$ are independent variables; they cause the beam deformation. In Fig. 7.3, on the contrary, the independent variables are displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$. This is achieved by installing the appropriate supporting bars. Supporting reactions that occur in them coincide with forces $P_1, P_2, \ldots, P_n$, but they are already the functions of set displacements.

To present potential energy as a function of displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ we express forces $P_i$ through these displacements. To do this, write $n$ equations of the form (7.2) and solve them in respect of the values $P_i$. We have no need to make this operation, only note that it is possible and that a solution will be $n$ formulas of the form:

$$P_i = a_{i1}\Delta_1 + a_{i2}\Delta_2 + \ldots + a_{in} \Delta_n. \quad (7.4)$$

If we put in this formula $\Delta_1 = 1$, $\Delta_2 = \Delta_3 = \ldots = \Delta_n = 0$, we obtain $P_i = a_{i1}$. It follows that the coefficient $a_{i1}$ expresses the value of the reaction that occurs in the $i$-th constraint, when the displacement equal to one will occur in the direction of the $i$-th constraint, while the rest of $n-1$ constraints remain motionless. Let us mark this reaction through $r_{i1}$. Then $a_{i1} = r_{i1}$ and in general $a_{ik} = r_{ik}$. This feature automatically follows the reciprocity of factors $a_{ik}$ that is $a_{ik} = a_{ki}$.

So:

$$P_i = r_{i1}\Delta_1 + r_{i2}\Delta_2 + \ldots + r_{in} \Delta_n, \quad (7.5)$$

then, according to the formula (7.2)

$$U(\Delta) = \frac{1}{2} \sum r_{i1}\Delta_i^2 + \sum r_{ik}\Delta_i \Delta_k, \quad i \neq k. \quad (7.6)$$

Such is the potential energy of elastic deformation.

The third form of potential energy we obtain, if we choose partially forces, partially displacements as independent variables. Let for some $m$ points the specified forces $P_1, P_2, \ldots, P_m$ acting on them, and for other points $n-m$ – displacements $\Delta_{m+1}, \Delta_{m+2}, \ldots, \Delta_n$. We take these values as independent variables. To do this, imagine that
constraints (e.g. support bars) are placed in the points \( m+1, m+2, \ldots, n \) and these constraints have displaced by the set values \( \Delta_m, \ldots, \Delta_n \). The reactions of these constraints will play the role of the forces \( P_{m+1}, P_{m+2}, \ldots, P_n \) and external forces will play the role of forces \( P_1, P_2, \ldots, P_m \) (Fig. 7.3, c).

Dependent \( m \) displacements and \( n - m \) forces can be expressed through the chosen independent variables by formulas

\[
\begin{align*}
P_i \ (\text{при } i > m) & = \sum_{k=1}^{k=m} r_i^k P_k + \sum_{k=m+1}^{k=n} r_i^k \Delta_k; \\
\Delta_i \ (\text{при } i \leq m) & = \sum_{k=1}^{k=m} \delta_i^k P_k + \sum_{k=m+1}^{k=n} \delta_i^k \Delta_k. \\
\end{align*}
\]

(7.7)

Here, \( \delta_i^k, \delta_i^k', r_i^k, r_i^k' \), mark displacements and reactions caused by the action of individual independent variables in the point \( i \), i.e. \( P_k = 1 \) or \( \Delta_k = 1 \).

Hence, it is easy to obtain an expression for the potential energy:

\[
U(P, \Delta) = \frac{1}{2} \sum_{i=1}^{m} P_i \Delta_i = \frac{1}{2} \sum_{i=1}^{m} \delta_i^2 P_i^2 + \sum_{i=1}^{m} \delta_i^k P_k + \\
+ \frac{1}{2} \sum_{m+1}^{n} r_i \Delta_i^2 + \sum_{m+1}^{n} r_i \Delta_k, \quad i \neq k.
\]

(7.8)

This formula for potential energy is called a *mixed* one.

It follows from this formula that, if the system fixed at the points \( m+1, m+2, \ldots, n \) is subjected to combined action of forces \( P_1, P_2, \ldots, P_m \) and displacements \( \Delta_m, \Delta_{m+2}, \ldots, \Delta_n \), the potential energy equals the sum of energies that would yield under separate action of these forces on the one hand, and displacements on the other hand. Terms expressing mutual work of these two factors are absent.

It is easy to understand, if you imagine that the displacements \( \Delta_m, \Delta_{m+2}, \ldots, \Delta_n \) first took place, and then appeared forces \( P_1, P_2, \ldots, P_m \). When the system deformation process occurs owing to
these forces, the above constraints do not perform additional
displacements, and therefore their reactions only change their value,
but do not perform additional work. The resulting conclusion can be
written as follows:
\[
U(P, \Delta) = U^{\text{comp}}(P) + U(\Delta). \tag{7.9}
\]
Assume that all the forces \( P_1, P_2, \ldots, P_n \) are independent
variables. Under this condition differentiate the both sides of formula
(7.2) by one of the forces, \( P_1 \) for example, and obtain
\[
\frac{\partial \Delta_i}{\partial P_1} = \delta_{i1} = \delta_{1i}. \tag{7.10}
\]
Then, making differentiation following the same variable formula
(7.1) and using the relation (7.10), we find
\[
\frac{\partial U(P, \Delta)}{\partial P_1} = \frac{1}{2} \left( \Delta_1 + P_1 \frac{\partial \Delta_1}{\partial P_1} + P_2 \frac{\partial \Delta_2}{\partial P_1} + \ldots + P_n \frac{\partial \Delta_n}{\partial P_1} \right) =
\]
\[
= \frac{1}{2} \left( \Delta_1 + P_1 \delta_{11} + P_2 \delta_{12} + \ldots + P_n \delta_{1n} \right) = \frac{1}{2} \left( \Delta_1 + \Delta_1 \right) = \Delta_1. \tag{7.11}
\]
The same result we could get after direct differentiation of
formula (7.3). Thus, on the above condition a partial derivative of
potential energy of one of the forces equals the displacement of the
point of this force application in the direction of the latter. This
theorem is known as Castigliano theorem

If there is no external force in the point, which displacement we
are looking for, then this force should be applied, lettered, and then,
the expression of potential energy should be derived and
differentiated with respect to this force. After performing these
operations, we obtain the expression for the desired displacement,
and it remains only to equate the introduced force to zero

The second derivative also has a simple physical meaning: it
follows from the formulas (7.10) and (7.11) that
\[
\frac{\partial^2 U(P, \Delta)}{\partial P_i^2} = \delta_{ii}. \tag{7.12}
\]
Hence it is seen that the second force derivative is always positive. Further:

$$\frac{\partial^2 U(P, \Delta)}{\partial P_i \partial P_k} = \delta_{ik}. \quad (7.13)$$

Sometimes in the structural mechanics we have to solve the inverse problem: to find appropriate force $P_1, P_2, \ldots, P_n$ from the data of total displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$. This problem occurs, for example, when the reactions of redundant supporting bars, caused by displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ of these bars, are the forces $P_1, P_2, \ldots, P_n$. In these cases the following property of potential energy, established by J.-L. Lagrange, is of interest: if displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ are considered as independent variables, the partial derivative of potential energy with respect to any of these displacements is equal to the appropriate force, i.e.

$$\frac{\partial U(\Delta)}{\partial \Delta_i} = P_i. \quad (7.14)$$

That may be proved similarly to the Castigliano theorem.

Consider the special case, when forced displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ are set at the system certain points, after that these points were fixed. The reactions $R_1, R_2, \ldots, R_n$ of corresponding constraints play the role of external forces, so from formula (7.14) we obtain

$$\frac{\partial U(\Delta)}{\partial \Delta_i} = R_i, \quad (7.15)$$

and from formula (7.6)

$$\frac{\partial^2 U(\Delta)}{\partial \Delta_i^2} = r_{ii}. \quad (7.16)$$

If we resort to the mixed expression of potential energy, we will note that it admits differentiation with respect to variables of both types. Differentiate formula (7.9) with respect to independent variables of forces $R_1, R_2, \ldots, R_m$. The expression $U(\Delta)$ does not
depend on these forces, so a derivative of the potential energy is such as if the potential energy is only a function of forces:

$$\frac{\partial U(P, \Delta)}{\partial P_i} = \frac{\partial U^\text{comp}(P)}{\partial P_i} = \Delta_i. \quad (7.17)$$

A partial derivative of potential energy expressed in a mixed form equals, with respect to one of independent forces $P_i$, to the total displacement $\Delta_i$ in the direction of this force, obtained provided that the points of application of force $P_{m+1}, P_{m+2}, \ldots, P_n$ are fixed.

Similarly,

$$\frac{\partial U(P, \Delta)}{\partial \Delta_i} = \frac{\partial U(\Delta)}{\partial \Delta_i} = R_i, \quad (7.18)$$

i.e. partial derivative of the same expression in respect of the set independent displacements $\Delta_i$ equals the reaction of the corresponding $i$-th constraint, which would appear at the given independent displacements $\Delta_{m+1}, \Delta_{m+2}, \ldots, \Delta_{m+n}$, if the external forces $P_1, P_2, \ldots, P_m$ were absent.

From formula (7.6) we get

$$\frac{\partial^2 U(\Delta)}{\partial \Delta_i \partial \Delta_k} = r_{ik}. \quad (7.19)$$

The presented dependencies can be also obtained from general considerations.

As is known [Bazhenov, 2014], Euler's theorem on homogeneous functions $f(x_1, x_2, \ldots, x_n)$ of k dimension evidences that

$$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \ldots + \frac{\partial f}{\partial x_n} x_n = kf(x_1, x_2, \ldots, x_n).$$

If $p_i = \frac{\partial f}{\partial x_i} x_i$ and det $\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \neq 0$

$$f(x_1, x_2, \ldots, x_n) + H(p_1, p_2, \ldots, p_n) = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n.$$  

Young's inequality

382
\[ f(x_1, x_2, \ldots, x_n) + H(p_1, p_2, \ldots, p_n) \geq \sum_{i=1}^{n} p_i x_i. \]

Legendre transform

\[ f(x_1, x_2, \ldots, x_n) + H(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i x_i. \]

\begin{align*}
\text{Lagrange theorem} & \quad \text{Castigliano theorem} \\
p_i &= \frac{\partial f}{\partial x_i} & x_i &= \frac{\partial H}{\partial p_i}.
\end{align*}

If the function is a quadratic form, then \( f(x) = x^T A x \), where \( A \) – matrix of quadratic forms. Since \( A \) – symmetric matrix, then \( A^T = A \). According to Euler's theorem, \( p^T x = 2 f(x) \) or \( p^T x = 2x^T A x \).

Hence \( p^T = 2x^T A \), \( x^T = \frac{1}{2} p^T A^{-1} \). Then

\[ H(p) = f(x) = \frac{1}{2} p^T A^{-1} \cdot A \cdot \frac{1}{2} A^{-1} \cdot p. \]

Thus the value of the quadratic form \( f(x) \) and its Legendre transform \( H(p) \) match in the respective points

\[ f(x) = H(p). \]

The Legendre transform in the case of a function of potential energy of elastic deformation \( U(\Delta) \) and complementary potential energy \( U^{\text{comp}}(\Delta) \) has the form

\[ P^T = \{ P_1, P_2, \ldots, P_n \}; \quad \Delta^T = \{ \Delta_1, \Delta_2, \ldots, \Delta_n \}. \]

Potential energy of elastic deformation

\[ U(\Delta) = \frac{1}{2} \Delta^T K \Delta. \]

Additional potential energy

\[ U^{\text{comp}}(P) = \frac{1}{2} P^T D P. \]

Young's inequality is

383
\[
\frac{1}{2} \Lambda^T K \Lambda + \frac{1}{2} P^T D P \geq P^T \Lambda.
\]

Work equality of internal and external forces is the Legendre transform

\[
\frac{1}{2} \Lambda^T K \Lambda + \frac{1}{2} P^T D P = P^T \Lambda
\]

The conditions of equilibrium, strain compatibility and boundary conditions should be fulfilled.

The condition that converts Young's inequality in equality is

\[
P = K \Delta \text{ or } \Lambda = D P.
\]

In so doing matrices \( K \) and \( D \), which represent, respectively, the rigidity and compliance matrices are mutually inverse: \( K D = E \). These matrices are matrices of second derivatives (Hessian matrices) of potential energy of elastic deformation and complementary potential energy, their coefficients equal:

\[
k_{ij} = \frac{\partial^2 U(\Lambda)}{\partial \Delta_i \partial \Delta_j}; \quad \delta_{ij} = \frac{\partial^2 U^{\text{comp}}(P)}{\partial P_i \partial P_j}.
\]

According to Donkin’s theorem, if two dual (by Young) functions of potential energy \( U(\Lambda) \) and \( U^{\text{comp}}(P) \) depend on the same parameter, say \( \eta \), or group of parameters that are not active, that is they take no part in the Legendre transformation, then there is the dependence

\[
\frac{\partial U(\Lambda)}{\partial \eta} = -\frac{\partial U^{\text{comp}}(P)}{\partial \eta}.
\]

For example

\[
U(\Delta) = \frac{1}{2} k \Delta^2, \quad U^{\text{comp}}(P) = \frac{1}{2} \frac{P^2}{k},
\]

\[
\frac{\partial U(\Delta)}{\partial k} = \frac{1}{2} \Delta^2, \quad \frac{\partial U^{\text{comp}}(P)}{\partial k} = -\frac{1}{2} \frac{P^2}{k^2} = -\frac{1}{2} \Delta^2.
\]

384
Relevant extreme problems for the Legendre transform give dual Lagrangian formulations of extreme problems (the Lagrange-Castigliano principles).

<table>
<thead>
<tr>
<th>Direct problem</th>
<th>Dual problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left{ \frac{1}{2} \Delta^T K \Delta - \bar{P}^T \Delta \right} \rightarrow \min ), on condition that ( \Delta = \bar{\Delta} ).</td>
<td>( \left{ -\frac{1}{2} P^T D P + \bar{\Delta}^T P \right} \rightarrow \max ), on condition that ( P = \bar{P} ).</td>
</tr>
</tbody>
</table>

### 7.2. On the history of the force method\(^{31}\) and displacement method

Great importance of the principle of possible displacements for mechanics of solids and solids system was fully appreciated after the appearance of *Analytical Mechanics* by Lagrange (1788).

The time of publication of the Clapeyron theorem on real work of elastic forces (1852) is to be considered the beginning of a broad energy trend in the development of the theory of elasticity and design of statically undeterminable systems.

The next significant step in the theory of elastic bodies consisted in the discovery of the principle of works reciprocity.

For the first time the reciprocity principle was derived in a purely algebraic form by A.L. Cauchy, who proved in 1857 the following property of any homogeneous quadratic function of several variables, if

\[ y_1 = F(\alpha_1, \beta_1, \gamma_1, \ldots, \theta_1), \quad y_2 = F(\alpha_2, \beta_2, \gamma_2, \ldots, \theta_2), \]

\[ \frac{\partial y_1}{\partial \alpha_1} \alpha_2 + \frac{\partial y_1}{\partial \beta_1} \beta_2 + \ldots = \frac{\partial y_2}{\partial \alpha_2} \alpha_1 + \frac{\partial y_2}{\partial \beta_2} \beta_1 + \ldots \]

But if we will understand \( y_1 \) and \( y_2 \) as potential energies corresponding to the two stress states, that is \((\alpha_1, \beta_1, \gamma_1, \ldots, \theta_1)\) and \((\alpha_2, \beta_2, \gamma_2, \ldots, \theta_2)\), the written above partial force derivatives will

---

\(^{31}\) A detailed historical essay on development of special techniques of force method is in the book: I.M. Rabinovich *Methods for Frame Analysis* [Rabinovich, 1934, pp. 10-25].

---

385
express the corresponding displacements, and work reciprocity theorem will become only a special case of the Cauchy theorem. But A.L. Cauchy himself, though he was too close to the theory of elasticity, has not made this conclusion from his theorem. Note the obvious connection of the presented theorem with well-known Euler's theorem for homogeneous functions (1779).

The property of reciprocity, which refers to an elastic body, was established in 1864 by J.K. Maxwell in the theorem on displacements reciprocity:

\[ \delta_{mn} = \delta_{nm}. \]

In the same paper J.K. Maxwell proved the following lemma for a truss: if under the action of the force "unit", applied between two nodes B and C, in a bar s the force \( S = p \) is obtained, the given bar being extended as \( \Delta s = 1 \), the extension of segment BC is also equal to \( p \). Using current notations, we can say that the ratio for such a system is obtained at \( P = 1 \)

\[ r_{1P} = -\delta'_{P1}. \]

The work reciprocity principle for any elastic body was proved in 1872 by E. Betty. He considered two states of a body that is under the joint influence of surface and volume forces, and expressing the work through stress and displacement he has shown that the factors related to the both states enter symmetrically in the virtual work expression. He also allowed for the work of inertia forces. Proceeding from his theorems he has made a number of important conclusions for the elasticity theory.

Almost simultaneously with E. Betty, but one-two years later, Rayleigh published a number of works devoted to the issues of reciprocity.

Rayleigh issued from much more common but simpler considerations than E. Betty. In his first articles he had not come to the end, and instead of proving reciprocity of works, confined himself to the proof of reciprocity of displacements and reactions.
In 1873 he proved that, if periodic force of harmonic type with a
given amplitude and period affects the system at the point \( A \), the
displacement amplitude and phase of another point \( B \) will be the
same as they could be at the point \( A \), if the force affected the point \( B \).
Next year he made a conclusion of static nature from this theorem,
suggesting that the force period is very large, and the effect of
dispersion and inertia forces is very small. Then at any instant the
body will have such configuration that it would have under the action
of the same forces that remain unchanged. Hence, he obtained three
static ratios that will be written in our notation as follows:
\[
\delta_{12} = \delta_{21}; \quad \eta_{12} = \eta_{21} \quad \text{and} \quad \delta'_{p2} = -\eta_{2p}.
\]

The second of these equations, according to Rayleigh, are also
extended to system vibrations. Despite this, in 1875 he proved
separately the theorem of displacement reciprocity for a beam with
any variable section, supported at the ends. Only in 1877 in the first
edition of his book on the theory of sound he gave the law of work
reciprocity in the form of the ratio
\[
\Psi_1 \psi_1' + \Psi_2 \psi_2' + \ldots = \Psi_1' \psi_1 + \Psi_2' \psi_2 + \ldots,
\]
where generalized forces are marked by \( \Psi \) and generalized
displacements by \( \psi \).

Purely algebraic, and therefore general interpretation of the
reciprocity principle was re-formulated in 1927 by P.L. Pasternak. He
considers reciprocity as a property inherent in any system of \( n \) linear
equations with \( n \) unknowns having a symmetric matrix (i.e.
symmetry of coefficients). Let the system has the form
\[
a_{11}Y_1 + a_{12}Y_2 + \ldots + a_{1n}Y_n = c_1;
\]
\[
a_{21}Y_1 + a_{22}Y_2 + \ldots + a_{2n}Y_n = c_2;
\]
\[
\vdots
\]
\[
a_{n1}Y_1 + a_{n2}Y_2 + \ldots + a_{nn}Y_n = c_n,
\]
where \( a_{ik} = a_{ki} \).

If you replace the right part of equations \( c_1, c_2, \ldots, c_n \), with new
\( c_1', c_2', \ldots, c_n' \), and the roots of the new system denote by

387
If the forces are $Y'_1, Y'_2, \ldots, Y'_n$, then the following relationship will always be satisfied

$$Y'_1 c'_1 + Y'_2 c'_2 + \ldots + Y'_n c'_n = Y'_1 c_1 + Y'_2 c_2 + \ldots + Y'_n c_n.$$  

The principle of work reciprocity is only a partial illustration of this general algebraic theorems, if we understand $Y_i$ and $Y'_i$ as two sets of forces (or displacements), and $c_i$ and $c'_i$ as the corresponding total displacements (or forces). But in order to use this method of reasoning, it is necessary to prove beforehand the reciprocity of factors, that is to prove a theorem $\delta_{ik} = \delta_{ki}$ or $r_{ik} = r_{ki}$.

Even after the reciprocity principle became known and was applied, there appeared a book by K.A. Castiglano. It was published in Italian in 1875 and in French in 1879. However, his book is also devoted to general properties of elastic body and, like the above works, is based on the properties of potential deformation energy.

I.M. Rabinovich [Rabinovich, 1954 (Volume 2)] notes that looking from our historical distance at that time, one cannot but wonder at the force with which scientific ideas were finding their way into life, when they were needed. The idea of energy method for studying an elastic body, a method that uses in different ways the work deformation properties that unites the works of all these authors. They, independently of each other in this era being so important for structural mechanics, laid the foundations of the modern theory of statically undeterminable systems.

As to the principle of the least work, it was advanced by P.L. Maupertuis as the universal, comprehensive natural law in the 18th century. But it could not be strictly proved. In the 19th century (1858) it was proposed in a more limited formulation by L.F. Menabrea, and his proof was insufficiently rigorous. Castiglano presented a satisfactory proof, valid for linearly deformable systems.

Energy ideas that enriched the structural mechanics found a brilliant interpretation in the work by V.L. Kirpichov Redundant Unknowns in Structural Mechanics, published in 1903. All the works by V.L. Kirpichov were exceptionally clear, demonstrative and generally stated. In these works he gave a brief but comprehensive
presentation of the work reciprocity principle, theorems on
derivatives of potential energy and on minimum of the work of
deformation. Application of these principles was illustrated by
examples. Small in volume but rich in meaning, the work by V.L.
Kirpichov involved Russian engineers in a range of progressive ideas
of structural mechanics of that time and exerted great influence on
the development of science as a whole in Russia.

Later on a systematic exposition of these principles in the course
of lectures for a higher school was given in a more modern form by
M.M. Filonenko-Borodich [Filonenko-Borodich, 1932]. One of the
first works devoted to the study of the properties of potential energy
was a research of M. Bekhterev, published in 1925 [Bekhterev 1925].
Based on the properties of potential energy to keep positiveness,
Bekhterev has derived some interesting inequalities between
constants that bind stresses and deformations of the anisotropic
elastic body. Some properties of potential energy that are significant
for the problems of stability and dynamics of structures (for example,
energy properties of composite systems formed by fastening of two
main systems) were indicated by Ya.L. Nudelman [Nudelman, 1949].
Ya.M. Rippenbeyn [Rippenbeyn 1927] extended the term of the work
of external and internal forces entering into consideration a new kind
of loads: "pairs of $n$-th order" with the dimension $kg \cdot m$.

O.A. Umansky has formulated the work reciprocity principle
[Umansky, 1935, p. 39], his formulation follows from the extended
idea of the two "states" of the system: it is assumed that certain final
deformations are given jointly with the external forces. For example,
breaks are set in some sections in the elastic line or its derivative.
When deriving the expression of reciprocal work, each force and
local deformation of one state is multiplied by the corresponding
displacement or corresponding internal force of the other state.

Energy theorems are also known to develop in the other
direction: the extending to the nonlinearly deformed body. The first
attempt in this direction was made by V.E. Novodvorsky (1889-
1933). The next significant work belongs to M.I. Bezukhov.

389
The method of solving undeterminable static problems, where the forces in redundant constraints are considered as unknowns (force method), appeared a long time ago. At least implicitly, it appeared in the analysis of continuous beams even in 1808. Canonical equations of the force method were first presented in the literature by J.K. Maxwell in 1864. He derived these equations for statically indeterminate truss, using the principle of possible displacements and considering subsequently possible work of forces $X_1 = 1, X_2 = 1, \ldots$ on real displacements of the system.

These equations were further developed in the 70's and 80's of the 19th century in the works by A. Mohr and a number of other authors. As was noted, V.L. Kirpichov with his brilliant and often original lecturing has made great contribution to the acquaintance of wide engineering circles with the general theory of analysis of statically indeterminate bar systems in the late 19th and early 20th centuries.

Special techniques of force method assigned to overcome the difficulties of analysis of statically undeterminable systems with the large number of unknowns, began developing a long time ago. The first result of these studies was the derivation of the three-moment equation for a continuous beam.

The idea of vanishing the coefficients of some canonical equations by introducing infinitely rigid bars and transfer of redundant unknowns to them were first applied in 1881 (A. Mohr), the idea of using the grouped unknowns appeared in 1892 (G. Müller-Breslau). A combination of redundant unknowns was initially used as the simplest grouping in symmetrical and inversely symmetrical groups, then appeared more complex groups of unknowns formed following the principle of linear homogeneous transformation using arbitrary coefficients.

In the 19th century calculation of statically indeterminable frames was developed by many authors. Initially, the main task was to calculate frames with parallel horizontal chords and vertical pillars (girder-free trusses as they were then called). Only later, when the frame skeletons of various kinds have been widely used in industrial and civil construction, the topic was extended.
One of the first works belongs to L.F. Nikolai, professor of St. Petersburg Institute of Railway Engineers [Nicolai, 1904]. The next significant research belongs to G.P. Perederiy [Perederiy, 1906]. He has derived the basic formulas for the frame, which has the form of a rectangular closed loop with different ratios of bars rigidity, and then extended them to multilayer girder-free one-panel frame with parallel chords.

In 1909 there appeared the original book by I.S. Podolsky [Podolsky, 1909], which contained a simple approximate calculation of girder-free frames with parallel and non-parallel chords of static and movable load.

In 1913 a work of M.S. Streletsky was published which concerned the method of calculating the same frames with parallel chords and nodal load.

At the same time, there was published the book that has a special place in the literature on the theory of statically indeterminate bar systems, a book by V. Bashynsky [Bashynsky, 1913]. His calculation of every frame system begins with the fact that the elastic line equation as algebraic polynomial is written for each straight bar; degree of the latter is 4 units above the degree of a curve that expresses the load intensity. Polynomial coefficients are determined then from the system of equations that express the deformation continuity at the intersection of bars and conditions for fixing by supports. The book by V. Bashynsky had useful ideological influence on the further development of the theory of elastic line. The book by M.S. Streletsky dedicated to calculation of complex statically indeterminate systems [Streletsky, 1921], which appeared in 1921, may be considered as the beginning of a new stage in the theory of frames.

First ideas of the displacement method occurred in the literature only as hints, which were not clear even to the authors. For example, in 1862 E. Winkler derived formulas to calculate a continuous beam; the bar bending moments in these formulas are expressed as functions of rotation angles of the bar ends and of a turn of the bar. But Winkler has not arrived to understanding that this method can be generalized and in further presentation he tried to get rid of these
variables by expressing them through moments. J.A.Ch. Bresse went somewhat further, and in 1865 derived a three-angle equation for a continuous beam. However, he then said: "it seems unnecessary to further develop these considerations, the practical use of which will be very limited." Apparently, he abandoned the development of ideas that could lead him to the discovery of the displacement method, and has not noticed their significance.

Besides the above works by E. Winkler (1862) and J.A.Ch. Bresse (1865) we should mention previously performed studies of D.I. Zhuravsky [Bernstein, 1967] on the design of trusses. He believed that the truss chord area between two “sections of loads division” (terminology of D.I. Zhuravsky) are in the same conditions as a bar with both ends fixed, the axial force being applied along its length. This problem, which has become a classic one, is statically undeterminable, and D.I. Zhuravsky solves it by the method of deformation.

![Image of historical figures]

**Mykhail Mytrofanovych Filinenko-Borodych** (1885-1962)  
**Emil Winkler** (1835-1888)  
**Jacques Antoine Charles Bresse** (1822-1883)  
**Dmitriy Ivanovich Zhuravsky** (1821-1891)

In generalizing article by A. V. Perelmutter *On the 100th Anniversary of the Displacement Method* [Perelmutter, 2014] it is stated that in the second half of the 19th century the force method has defined the face of the classical structural mechanics. Today this role belongs to the displacement method – one of the most important strongholds of modern structural mechanics, which underlies practically all the computer codes. The internal structure of this method proved ideal for formalized approach oriented to
implementation by computers. The displacement method for the frame design was developed in 1914 by Danish engineer Axel Bendixsen [Bendixsen, 1914].

Seven years later, Asher Ostenfeld, Professor of Copenhagen Technical University, presented equations for displacements in the same form as the equations for the force method forces that were then already known. A. Ostenfeld rejected kinematic approach by A. Bendixsena and assumed the equilibrium equations of nodes as a basis of reasoning, he introduced in the equations the reactions of bars, which converge in the node, to the node single turn. Previously studied reactions of individual bar elements of the system were, according to A. Ostenfeld, those "bricks" that do not allow starting each analysis from the very beginning and, as is clear now, have become the prototypes of modern finite elements. A. Ostenfeld has introduced the term "displacement method" and pointed to its formal duality with the force method [Ostenfeld, 1921].

In 1927 almost simultaneously Ludwig Mann [Mann, 1927] and A.A. Gvozdiov [Gvozdiov, 1927], based on the classical equations of the second kind in Lagrange analytical mechanics, imparted the final form to the displacement method that has survived up to this day. In particular, a clear description of the main system of the displacement method is given, and properties of the factors of the canonical system of equations and groupings of unknowns are stated in the book by A.A. Gvozdiov.

A fundamental work of A. Bendixsen was based on some previous works. For example, A. Clebsch in 1862 in his course of elasticity theory, which included the theory of bar systems, wrote: "... we will consider the displacement of nodes as previously known parameters to determine from them the elastic forces, to which the bars react in their corners and finally establish equilibrium conditions for the external and elastic forces acting in nodes; then these equations will allow calculating the displacements » [Clebsch, 1883].

Following the works by L. Mann and A.A. Gvozdev, the displacement method developed in the following directions:
the connection between the force method and displacement method was analyzed in more detail, in 1934 G.Kruck [Kruck, 1934] proposed a variant of the method with a complex basic system, where not individual rods but rod subsystems were used as the “building material”, i.e., essentially the ideological foundations of the superelement method have been laid;

- the dual nature of the main methods of structural mechanics was studied in the works of P. Pasternak [Pasternak, 1922] and A. Hertwig [Hertwig, 1933], while A. Schlesusner analyzed their relationship with the variational approaches [Schlesusner, 1933];

- E. Fliegel expanded the scope of the displacement method application to stability problems [Fliegel, 1938], and V. Koloushek – the problems of dynamics [Kolousek, 1941].

A notable milestone in the development of the displacement method was the work by J.M. Rippenbeyn [Rippenbeyn, 1933]. Obviously, he was the first to apply the displacement method to spatial bar systems. Later in the monograph by D.V. Weinberg and V.G. Chudnovsky [Weinberg, Chudnovsky, 1948] a spatial problem of the displacement method was presented in a tensor form that was later successfully used in the theory of elasticity and theory of shells and served as one of the ways of integration of structural mechanics and the theory of continuous media.

The next fundamental step in development of the displacement method was the transition to the matrix formulation, developed in the
50's by George Argyris [Argyris, 1954, 1957], that further essentially led to the transition to the finite element method – a certain winner of this historical race.

Three most important events of the scientific revolution in structural mechanics occurred during the 20th century – matrix reformulation matrix of calculation algebra in structural mechanics [Argyris, 1954], the invention of the concept of finite element [Turner et al., 1956] and direct stiffness method, which is further development of the displacement method [Turner, 1959]. They broke through the chain of classical fundamental disciplines of engineering sciences and radically changed the design technologies in a number of new fields.

The calculation problems for the truss with rigid nodes also attracted attention of engineers in Russia. One of the first works was a book by E.O. Paton [Paton, 1901], which was published in 1901. It includes a review of all published by the time methods of calculation, and, using one of them, the extensive research was carried out aimed at finding the relative increment of stresses arising in trusses due to node rigidity. All the methods presented in the book by E.O. Paton are essentially different versions of the displacement method.

In 1907, N.V. Nekrasov published a book [Nekrasov, 1907]; the diagram of the exact solution of the problem given in the book is free of simplifying assumptions that were made by his predecessors. To
solve a system of equations with many unknowns that inevitably arises with this solution, the author offered the Gaussian method, which is used for balancing errors by the method of least squares. In 1909 S.I. Belzetsky [Belzetsky, 1909] proposed a simpler but most approximate solution of the problem. It is presented in a general form that can be used for any truss. The number of unknown rotation angles in his book was equal to \( k-2 \), where \( k \) is the number of truss nodes. We should note theoretical and experimental studies of K.M. Dubyaha [Dubyaha, 1914], who also presented a critical review of Russian and foreign literature.

Further the displacement method was comprehensively developed. The concepts of simple and group reactions caused by elastic displacements, temperature and load were developed, and formulas for them derived; this method equations were presented in canonical form and in expanded form; the ways to simplify equations and to use symmetry were developed as well. Method of calculating the combined trusses was elaborated. A.A. Gvozdiov has proposed a mixed method that is synthesis of the force method and displacement method. The displacement method obtained a completed form of the classical method. A.A. Gvozdiov, P.L. Pasternak, V.N. Zhemochkin, M.I. Pierre, I.M. Rabinovich, A. Rabtsevych and others took part in its development.

### 7.3. Matrix formulation. Argyris

Development of the displacements method and associated determination of dual nature of structural theory provided most powerful advancement of knowledge in this fundamental building theoretical discipline during its consolidation. The general theorem of work (the work equality of internal and external forces), which was successfully applied by A. Mohr since 1874 for girder systems has a dual structure: the principle of possible changes in the stress state (the force method basis) and the principle of possible displacements (displacement method basis, Fig. 13.9) [Kurrer, 2008].

In the theory of statically undeterminable systems and practical
structure analysis the force method, based on the principle of possible changes of the stress state, quickly began to dominate in the direct or indirect form of Castigliano theorems (second Castigliano theorem). This was conditioned by the general tendency to formalization of structure analysis, which was embodied in the $\delta$-symbols. Although A. Mohr and his student Robert Lund had recognized the fundamental role of the principle of possible displacements, this method remained on the background. One of the reasons was, of course, that the formulations based on equilibrium conditions were preferred as more familiar to engineers. However, $\delta$ – characters, are closely related to the force method, provided for the displacement method in formal terms and favored its advancement.

Note that the displacement method is sometimes mentioned as elasticity equation of type 2. L. Mann first used the principle of possible displacements to determine forces in constraints. Formal development of the displacement method followed the force method. The dual nature became apparent as early as 1902, when appeared the already mentioned book by V.L. Kirpichov Redundant Unknowns in Structural Mechanics. V.L. Kirpichov developed the theoretical foundations of both the force method and the displacement method. The second edition of this outwardly unpretentious but very meaningful work was published 21 years after the death of its author.

Unfortunately, the book by V.L. Kirpichov has not appeared in English, French or German. In the west it was ignored because of ignorance of the Russian language. Just the launch of the first artificial satellite (1957) favored a gradual recognition in the west of scientific progress achieved in Russia. One example of this recognition was the publication of the book by I. Rabinovich Structural Mechanics in the USSR 1917-1957 in English translated by J. Herrmann. In this book I.N. Rabinovich among other things reviews the most important publications on the force method and displacement method, which appeared in the Soviet Union for 40 years. Outstanding in this respect is the monograph by A.A. Gvozdiov, dated 1927, since it was first in Russian technical literature that the displacement method was presented in full amount.
Before 1933 J.M. Rippenbeyn laid the foundation of the three-dimensional displacement method, which three years later B.M. Gorbunov and Yu.V. Krotov presented, using tensor algebra, in the form of "motor symbolism,"\textsuperscript{32,33} Richard von Mises \textsuperscript{34}. Formalization

\textsuperscript{32} Following the concept of "motor", introduced by E. Study in 1903, mathematician Richard von Mises developed the mathematical aid for mechanicians in the form of "motor symbols."

\textsuperscript{33} Study, Eduard (1862-1930) – German mathematician, geometrist, took part in the development of symbolic notation in the theory of invariants.

\textsuperscript{34} Richard Edler von Mises (1883-1953) – mathematician and engineer of Austrian origin. Born in the city of Lemberg, Austria-Hungary (now Lviv, Ukraine). Major works are devoted to aerodynamics, applied mechanics, mechanics of fluids, aeronautics, statistics and probability theory. Investigated the stability of cylindrical shells, introduced the "motor symbols".

398
of the displacement method was continued by D.V. Weinberg and V.G. Chudnovsky. The book, published in 1948, included a three-dimensional displacement method in the tensor form. This work also reflects the importance of computational aspects of the theory of structure analysis.

In 1938 there appeared the work by Arno Shleusner, who, using the apparatus of variational calculus has clearly showed the conceptual difference between the principle of possible changes in the stress state and the principle of possible displacements with small displacements.

Works by G. Pranhe (1919) completed the first use of formal theory in the whole structural mechanics based on the calculus of variations. The second use of formalized theory in mechanics of structures was achieved by J.H. Argyris on the basis of matrix algebra. Both theories have a dual nature. ITU – a fusion of Structural Mechanics and Computational Mathematics at the transition from the innovation phase to the phase of distribution.

Content of a feeling of Johann Wolfgang von Goethe (1749 1832) for Immanuel Kant (1721 1804) which is described in a conversation with young Arthur Schopenhauer (1788 1860), can be referred to both works: "When I read Kant, I had a feeling as if I entered a sparkling room." J. Argyris had been read, and G. Pranhe mostly not. Much later, in 1942, A. Schlesner, Marhuer, Hamel, Hrammel, Klotter offered to publish a qualifying work by Pranhe, but failed. This work, which included an attempt to substantiate the theory of elasticity using variational methods was published only in 1999.

The first ideas of using matrices to analyze the structures were expressed by Eduard Study in 1903.

In 1957 John (Ioannis) Haji Argyris presented a matrix formulation of the theory of statically undeterninable systems and
formulated the theory of structure analysis in the matrix form. Describing the motives of his work, J. Argyris noted that none of the conventional static methods can be effectively used to determine the stress fields and compliance matrix of highly statically undeterminable systems of modern aircraft structures. Similar difficulties occur in other additions of statics. The iterative methods may be of use in some cases, but in general they are too labor-consuming to calculate bearing aircraft structures of the membrane and shell type. It proved possible to overcome these difficulties only using matrix formulation of statics, based on the use of electronic digital computers.

Matrix formulation not only allows calculating in the most clear way, but is also ideal form of recording for digital computers. Matrix theory calculations are so clear and understandable that new, practically valuable relationships, which in normal recording would be impossible and inaccessible are now easily obtained.
For the structure analysis we need now only three basic simple matrices plus load vector-column. In addition, analysis in a matrix form allows also solving the non-linear elastic and dynamic problems. J. Argyris has formed a practical dictionary of terms:

<table>
<thead>
<tr>
<th>Force method</th>
<th>Displacement method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forces</td>
<td>Displacements</td>
</tr>
<tr>
<td>Stress</td>
<td>Deformations</td>
</tr>
<tr>
<td>Internal forces</td>
<td>Node displacement</td>
</tr>
<tr>
<td>Compliance = Displacement: Force</td>
<td>Stiffness = Force: Displacement</td>
</tr>
<tr>
<td>Unit-load method</td>
<td>Unit-displacement method</td>
</tr>
<tr>
<td>Statically determinable system</td>
<td>Kinematically determinable system</td>
</tr>
<tr>
<td>Statically undeterminable system</td>
<td>Kinematically undeterminable system</td>
</tr>
<tr>
<td>Compliance matrix</td>
<td>Stiffness matrix</td>
</tr>
<tr>
<td>Generalized forces</td>
<td>Generalized displacements</td>
</tr>
</tbody>
</table>

and showed in matrix formulation the dual nature of the structural theory (Fig.7.5).

J. Argyris thus succeeded in converting structural analysis into the completed formalized theory; he summarized the results of its pioneering series of articles in 1960 in common with S. Kelsey.
monograph *Energy Theorem and Structural Analysis* [Argyris, Kelsey, 1960]. Describing the history of the finite element method, R.V. Clough very properly appraised this monograph as the most important work ever written on the theory of structural analysis (see essay by J. Argyris' *Computer Forms the Theory* [Argyris, 1965]).

<table>
<thead>
<tr>
<th>Force method</th>
<th>Displacement method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force $R$</td>
<td>Displacement $r$</td>
</tr>
<tr>
<td>✶ Compliance $F$</td>
<td>✶ Stiffness $K$</td>
</tr>
<tr>
<td>✶ Displacement $r$</td>
<td>✶ Force $R$</td>
</tr>
</tbody>
</table>

Generalized force $Q$

- $R = BQ$

Generalized compliance $F$

- $F_q = B'FB$

Generalized displacement $q$

- $q = B'r = F_qQ$

Generalized sequential assemblage

Element stress $S$

- $S = bR$

Element deformation $v$

- $r = b'v$

Element compliance $f$

(of stress $S$)

The whole structure compliance

- $F = b'fb$

It is always possible to change $a, b$ respectively, by $\bar{a}', \bar{b}'$

Element deformation $v$

- $v = ar$

Element stress $S$

- $R = \bar{a}S$

Element stiffness $k$

(of deformation $v$)

The whole structure stiffness

- $K = \bar{a}'ka$

Generalizes parallel assemblage

- $A'B' = I = B'A$

- $F_qK_q = I = K_qF_q$

- $K_q = A'KA$

- $Q = A'R = K_qq$
<table>
<thead>
<tr>
<th>Compliance addition</th>
<th>Stiffness addition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Special sequential assemblage)</td>
<td>(Special parallel assemblage)</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td><img src="image1" alt="Stiff" /></td>
<td><img src="image2" alt="Stiff" /></td>
</tr>
<tr>
<td>$F_a$ + $F_b$ = $F$</td>
<td>$K_a$ + $K_b$ = $K$</td>
</tr>
</tbody>
</table>

Fig. 7.5.

References

Abovsky, N.P. and Andreyev, N.P. (1973), *Vriatsionnyie printsipy teorii uprugosti i teorii obolochek* [Variational principles of elasticity theory and theory of shells], Krasnoyarsk, USSR.


Abrikosov, A.A. (1965), *Akademik L.D. Landau: korotkaya biography i obzor nauchnykh rabot* [Academisian L.D. Landau: brief biography and survey of scientific works], Nauka, Moscow, USSR.


Aleksandrov, A.V., Lashchennikov, B.Ya., Shaposhnikov, N.N. and Smirnov, V.A. (1976), *Metody raschota sterzhnevykh system, plastin i obolochek s ispolzovaniyem ECVM* [Methods of analysis of bar systems and shells with using EDC], Stroyizdat, Moscow, USSR.

Aleksandrov, A. V., Potapov, V.D. (1990), *Osnovy teorii uprugosti i plasticchnosti* [Foundations of the elasticity and plasticity theory], Moscow, USSR.


Bashinsky, N.N. (1913, 1930), Novy metod raschota balok i zhestkikh ramnykh system [New method of calculation of beams and rigid skeleton structures], Kiev-Moscow, USSR.


Wainberg, D.V. and Chudnovsky, V.G. (1948), Prostranstvennie ramnyie karkasy inzhenernykh sooruzheniy [Spatial frameworks of engineering structures], Gostekhizdat Ukrainy, Kyiv, USSR.


Gvozdiov, A.A. (1927), Oblastny metod raschota staticheski neopredelimykh system. Teoriya i primery yeyo primeneniya k raschetu ramnykh konstruktsiy [General method of analysis of statically undeterminable systems. Theory and examples of its application to frame structure analysis], MIIT, Moscow, USSR.

Dubyaga, K.M. (1914), Raschoty i ispytaniya raskosnykh ferm s zhestkimi soedineniyami v uzlakh [Calculations and tests of braced trusses with rigid joints in nodes], St.Petersburg, Russia.

Duveau, J. and Lions, J.-L. (1981), Neravenstva v mekhanike i fizike [Inequalities in mechanics and physics], Mir, Moscow, USSR.


404
Zenkevich, O. (1975), Metod konechnykh elementov v tekhnike [Method of finite elements in technology], Mir, Moscow, USSR.

Kirpichov, V.L. (1903), Lishniye neizvestnyie v stroitelnoi mekhanike. Raschet staticheski neopredelimykh system [Redundant unknowns in structural mechanics. Analysis of statically undeterminable systems], Izdatelstvo Kulzhenko, Kiev, Russia.

Kirpichov, V.L. (1917), Sobranye sochineniy. T. 1 [Collected works. Vol. 1], PPI, Petrograd, USSR.

Kirpichov, V.L. (1933), Osnovaniya graficheskoi statiki [Foundations of graphical statics], GTTI, Moscow, USSR.

Kirpichov, V.L. (1934), Lishniye neizvestnyie v stroitelnoi mekhanike [Redundant unknowns in structural mechanics], Gostekhteorizdat, Moscow-Leningrad, USSR.

Kirpichov, V.L. (1934), Lishniye neizvestnyie v stroitelnoi mekhanike: raschet staticheski neopredelimykh system [Redundant unknowns in structural mechanics: analysis of statically undeterminable systems], Moscow, USSR.

Kirpichov, V.L. (1951), Besedy o mekhanike [Discussions on mechanics], GITTL, Moscow-Leningrad, USSR.

Clough, B. and Pensien, J. (1979), Dinamika sooruzheniy [Dynamics of structures], Stroyizdat, Moscow, USSR.

Klein, F. (1937), Lektii o razvitiu matematiki v XIX stoletii [Lectures on mechanics development in the 19th century], ONTI, Moscow-Leningrad, USSR.

Kudryavtsev, P.S. (1974), Kurs istorii fiziki [Course of the history of physics], Prosveshchenie, Moscow, USSR.

Mikhlin, S.G. (1966), Chislennaya realizatsiya variatsionnykh metodov [Numerical implementation of variational methods], Nauka, Moscow, USSR.

Natsionalna Akademiya nauk Ukrainy [National Academy of Sciences of Ukraine], (1998), Fenix, Kyiv, Ukraine.

Nekrasov, N.V. (1907), K teorii ferm s zhestkim soedineniyami v uzlakh [On the theory of trusses with rigid joints in nodes], St.Petersburg, Russia.

Novozhilov, V.V. (1958), Teoriya uprugosti [Elasticity theory], Sudpromgiz, Leningrad, USSR.

Nudelman, Ya.L. (1949), *Metody opredeleniya sobstvennykh chastot i kriticheskikh sil dlya sterzhnevykh sistem* [Methods of determining eigen frequencies and critical forces for bar systems].

Papkovich, P.F. (1939), *Teoriya uprugosti* [Theory of elasticity], Oborongiz, Leningrad, USSR.

Paton, E.O. (1901), *Raschet skvoznykh ferm s zhestkimi uzlami* (Calculation of through trusses with rigid nodes), Moscow, Russia.

Paton, E.O. (1955), *Vospominaniya* [Memoirs], Goslitizdat Ukrayni, Kiev, USSR.

Paton, E.O. (1956), *Vospominaniya* [Memoirs], Derzhavne vydavnytstvo khudozhyoi literatury, Kyiv, USSR.

Perederiy, G.P. (1906), *K teorii bezraskosnykh ferm* [On the theory of braceless trusses], Moscow, Russia.


Perelmutter, A.V. (2011), *Ocherki po istorii metallicheskikh konstruktsiy* [Essays in the history of metal structures], Izdatelstvo Assotsiatsii stroielynnykh vuzov, Moscow, Russia.

Perelmutter, A.V. (2011), *Upravleniye povedeniyem nesushchikh konstruktsiy* [Control of behavior of carrying structures], Izdatelstvo Assotsiatsii stroielynnykh vuzov, Moscow, Russia.


Pisarenko, G.S. (1990), *Bibliografiya vchenyh Ukrainskoi RSR* [Bibliography of scientists of the Ukrainian SSR], Naukova dumka, Kyiv, Ukr.SSR.

Pisarenko, G.S. (1991), Stepan Prokopovykh Timoshenko, Nauka, Moscow, USSR.

Podolsky, I.S. (1909), *Bezraskosnye fermy. Ikh raschet i upravlenie k metallicheskim zhelezo betonnym konstruktsiyam* [Braceless trusses. Their calculation and application to metal and reinforced concrete structures], Izdatelstvo Instituta inzhenerov putei soobshcheniya, Moscow, Russia.


sistemy [Course of structural mechanics of bar systems. Vol. 1. Statically determinable systems. Vol. 2. Statically undeterminable systems], Gosudarstvennoe izdatelstvo po stroitelstvu i arkhitekturke, Moscow, USSR.

Rabinovich, I.M. (1984), Vospominaniya [Memoirs], Nauka, Moscow, USSR.

Rzhanitsin, A.R. (1956), Predisvayienie sploshnogo izotropnogo uprugogo tela v vide sharnirno-sterzhnevoi sistemy. Issledovaniya po stroitel’ny mekhanike i teorii plastichnosti [Presentation of continuous isotropic elastic body as a hinge-bar system. Studies in structural mechanics and elasticity theory], Gosstroyizdat Moscow, USSR.

Rzhanitsin, A.R. (1982), Stroitel’nyaya mekhanika [Structural mechanics], Vysshaya shkola, Moscow, USSR.


Rippenbeym, Yu.M. (1933), K raschotu ploskikh i prostranstvennykh statichskikh neopredel’nimykh system [On analysis of plane and spatial statically undeterminable systems], Gosstroyizdat, Moscow, USSR.


Spravochnik po teorii uprugosti [Reference book in elasticity theory], eds P.M.Varvak and A.F.Ryabov, Kiev, 1971, USSR.

Streletsky, N.S. (1921), K raschotu slozhnykh statichskikh neopredel’nymykh system [On analysis of complex statically undeterminable systems], Moscow, USSR.


407

Timoshenko, S.P. (1946), *Ustoichivost uprugikh sistem* [Stability of elastic systems], Translated from English, OGIZ, Moscow USSR.

Timoshenko, S.P. (1948), *Plastinki i obolochki* [Plates and shells], Translated from English, Gostekhizdat, Moscow USSR.

Timoshenko, S.P. (1957), *Istoriya nauki o soprotivlenii materialov s kratkimi svedeniyami iz istorii teorii uprugosti i teorii sooruzhenyi* [The history of science on material strength with brief data from the history of elasticity theory and structural theory], Gosteorizdat, Moscow USSR.

Timoshenko, S.P. (1912), *Kurs soprotivleniya materialov* [Course of materials strength], Izdatelstvo knizhnyi magazine Lidzikivskogo, Kiev, Russia.


Timoshenko, S.P. (1975), *Prochnost i kolebaniya elementov konstruktsiy* [Strength and vibrations of structure elements], Nauka, Moscow, USSR.

Timoshenko, S.P. (1975), *Staticheskie i dinamicheskie problemy teorii uprugosti* [Statrical and dynamical problems of the elasticity theory], Naukova dumka, Kiev, USSR.


Timoshenko, S.P. and Gere, J. (1976), *Mekhanika materialov* [Mechanics of materials], Mir, Moscow, USSR.

Timoshenko, S.P. and Gudier, J. (1979), *Teoriya uprugosti* [Elasticity theory], Nauka, Moscow, USSR.


Feodosyev, V.I. (1975), *Desyat lektstv-besed po soprotivleniyu materialov* [Ten lectures-disputes on material strength], Ed. 2, Nauka, Moscow, USSR.

Filin, A.P. (2007), *Ocherki ob uchonykh mekhanikakh* [Essays about scientists mechanicians], Izdatelsky dom Strategiya, Moscow, Russia.


Arogyris, J.H. (1968), Some new elements for the matrix displacement method, PN.


Appell, P. (1900), Traité de Mécanique, III.


Hertwig, A., (1933), Das “Kraftgrosenverfahren” und das “Formanderungsgrosenverfahren” für die Berechnung statisch unbestimmter Gebilde. *Der Stahlbau*, vol. 6, No. 19, pp. 145–149.


Kurrer, K.-E. (2008), The history of the theory of structures, Ernst & Sohn Verlag für Architektur und technische Wissenschaften GmbH & Co. KG, Berlin, Germany.


FRAGMENTS OF THE HISTORY
OF FINITE ELEMENT METHOD
Divide et impera.  

Louis XI

This simple idea to divide the body and control its individual parts, was very fruitful, it surpassed the efficiency of many deep and subtle thoughts, just canceled a series of previously recognized methods and buried for practice some theoretical constructs of analysts.

M.I. Reitman
It is difficult for today’s engineer to imagine the calculation of even not very complex structural system without using a computer program, their overwhelming majority being based on the finite element method (FEM). Surprisingly, but this method appeared relatively recently, and its development has taken less than three decades. During this time, the FEM has stepped far beyond its cradle (structural mechanics), and subject to its application was almost the whole field of scientific and technical problems described by equations in partial derivatives.

This proceeded in parallel with the development of computer technology, accompanied by scientific discussion, competition of developers, inevitable errors and unexpected discoveries. This essay is dedicated to some elements of the history of formation and development of FEM within the problems of structural mechanics.

**8.1. Prehistory**

Finite element method, an absolute favorite of modern structural mechanics (of our time), appeared in the middle of the 20th century and was the rightful heir to a number of earlier ideas that were born in the depths of classical structural mechanics – the fundamental ideas of the type of variational principles and purely technological ideas brought to life by engineering practice. They acted as precursors (perhaps it would be said – precursors) of some essential features of the finite element method.

**8.1.1. Physical discretization**

The transition from continual design model to some its close discrete analog (or continual scheme of smaller dimension) is an old tradition in the mechanics of deformable rigid body. Suffice it to recall, for example, that even Euler compared a membrane with a set of strings stretched in two mutually perpendicular directions [Euler, 1789]
In 1868 Gustav Kirsch managed to obtain the basic equations for a homogeneous, isotropic and linear elastic body from the bar model. Kirsch considers a system of points "connected by elastic constraints" to study the question, "whether it is possible to create an elastic system of points having the nature of isotropic medium, if the number of points is infinite" [Kirsch, 1868]. Rather than to obtain equilibrium conditions for an infinitesimal cube cut out of continuum Kirsch offers a linear-elastic space truss, consisting of 12 bars of a perimeter, four diagonal bars and eight spherical hinges (Fig. 8.1).

In this case Kirsch said that two elastic Lame constants
\[
\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = G = \frac{E \nu}{2(1 + \nu)}
\]
are equal to each other that corresponds to the value of Poisson's ratio \(\nu = 0,25\).

Thirty-nine years later in 1927 W. Riedel [Riedel, 1927] returned to the reverse operation of two-dimensional elastic continuum sampling in calculating the elastic stress state of elastic disk loaded with two families of absolutely rigid rods, with the help of equivalent truss based on quadratic rod cells. Fig. 8.2 shows an undeformed state (dotted line) of equivalent truss and its deformed final position in compression caused by the convergence of the top and bottom rods.

But this idea was proposed as a general working analytical method only in 1941 by A. Hrennikoff [Hrennikoff, 1941]. He calculated the values of truss element cross-sections, provided that the nodal displacements of brace model with identical boundary converge with the displacements of angular points of removable solid object.
Hrennikoff performed such calculations for the following rod cell: cuboid, equilateral triangle, rectangle and square. He studied, for example, elastic plates using quadratic rod cells. Moreover, he modeled an elastic plate as beam network. A. Hrennikoff thus finished the transition of modeling in structure analysis from a plane plate and shell structures through brace frame systems to trusses.

Later A. Hrennikoff also used the net method for linear analysis of stability loss of rectangular plates and developed trapezoidal rod cells for design of elastic plates.

![Diagram](image)

Fig. 8.2. Riedel scheme

Considerations of A. Hrennikoff boiled down to the following: if we form finite difference equations for the rod system and by passage to the limit, associated with characteristic cell size approaching the zero, we obtain differential equations that coincide with known differential equations that describe the behavior of continual systems, then the rod model taken as a basis can also serve as a model problem at finite sizes of the cell.

Such arguments were used in other studies. Various models of regular rod lattices simulating the behavior of the elastic continuum

417
were constructed. For example, E.R. Rzanitsyn in his work [Rzanitsyn, 1956] studies two variants of space trusses that can be considered as discrete analogs of the isotropic elastic body. In the first variant, the grid of space truss is the cube with the face diagonals, but without the cube diagonals, the cube diagonals being present in the second variant.

Bar approximation shown in Fig. 8.3 is used for the plane problem. Comparing the equilibrium equations arising in a given elongation (Fig. 8.3, b) and displacement (Fig. 8.3,c) with relevant equations of elasticity theory, A.R. Rzanitsyn obtained a connection between discrete model parameters and elastic constants of isotropic elastic body in the following form:

$$E = \frac{E_1 F_1}{a} \cdot \frac{2\eta + \sqrt{2}}{\eta + \sqrt{2}}, \quad \nu = \frac{\eta}{\eta + \sqrt{2}}, \quad \eta = \frac{E_2 F_2}{E_1 F_1}.$$  

\[\text{Fig. 8.3. A.R. Rzanitsyn model}\]

Another approach was used by the following authors [Dluhach, 1956], [Muzychenko, 1961], which justified a discrete model by coincidence of its equilibrium equations with finite difference presentation of differential equations of continuum. Thus, the biharmonic equation for the plane problem recorded through Erie’s function $\varphi(x,y)$, using the grid shown in Fig. 8.4,a, typical difference equation looks as follows:
\[20\varphi_{0,0} - 8(\varphi_{1,0} + \varphi_{-1,0} + \varphi_{0,1} + \varphi_{0,-1}) + 2(\varphi_{1,1} + \varphi_{-1,1} + \varphi_{1,-1} + \varphi_{-1,-1}) + \varphi_{2,0} + \varphi_{-2,0} + \varphi_{0,2} + \varphi_{0,-2} = 0.\]

This equation can be interpreted as a canonical equation of the force method formed for statically undeterminable truss which structure is shown in Fig. 8.4.b, if the angle \(\alpha = 0\), horizontal and vertical elements have the area \(F\), elements with an angle \(\alpha\) have the area of \(2F\), and the elements are oriented along the diagonals – the area \(2F\sqrt{2}\).

Fig. 8.4. Modeling of finite differences equation

In the above works repeated cells are the prototype of finite elements and differ from them only in that such cells are based on the discrete element (rod), while all structural FEM models are based on continual finite elements.

8.1.2. Displacement method

Perhaps the first who consistently used the displacement as a major unknown problem and built on this principle a course of elasticity theory was A. Clebsch. In 1862 he published a textbook on the elasticity theory [Clebsch, 1862], in which he applied the displacement equation not only to continual problems, but also to the elastic brace systems. Clebsch describes the basic idea of the displacement method that consists in computing displacements of nodes in elastic brace systems as follows:
"The general principle is here that the node displacements should be initially treated as known parameters to determine through them elastic forces with which the rods affect their nodes, and finally to establish equilibrium conditions for the external and elastic forces acting in nodes, then these equations allow calculating the displacements" [Clebsch, 1862, p. 413].

But the displacement method to analyze elastic brace systems in Clebsch formulation was rejected by engineers because then the theory of brace systems gave more attention to forces in the elements than the node displacements, and it was much easier to obtain solution for statically determinable hinge trusses, using the methods of graphical analysis.

Truss elements that converge in riveted joints not only are subjected to axial tension-compression forces, but also to bending moments; the latter cause additional bending stresses. Determination of the values of additional stresses is associated with a rather cumbersome calculations because the design systems of brace structures with rigid nodes have a high degree of static indeterminability.

In 1878 Henry Manderla presented a solution of the stress problem arising in the elements of the girder truss as a result of angle changes in truss joints caused by load. The Manderla solution [Manderla, 1880] allowed calculating additional stresses in simple trusses with rigid nodes based on the second order theory. In fact, after Clebsch had first introduced unknown parameters of displacements in the mathematical theory of elasticity for the calculation of through systems in 1862, Manderla did the same in 1879, in the theory of structures.

Following the train of thought of Manderla, E.Winkler introduces change of inclination angles of the end tangents to the node zones, that results in $k$ linear equations for $k$ nodes of $k$ moment equilibrium conditions [Winkler, 1881]. Unlike Manderla E.Winkler uses the
first-order theory for moments in the elements. Otto Mohr has put the final piece in the combined picture of the theory of added stresses. It was in 1892/93 [Mohr, 1892, 1893].

Mohr’s decisive contribution consisted in the clear division of node rotation angles $\phi$ and elements bias angles $\psi$ for unambiguous determination of the strain state of structures with rigid nodes.

In 1914 Danish engineer Axel Bendixsen extended Mohr’s approach to fixed and flexible frames [Bendixsen, 1914]. For fixed frames, assuming the absence of node displacement, Bendixsen uses the well-known expression for the bending moment $M_{mn}$ acting on the end $m$ of the rod $mn$

$$M_{mn} = \frac{2EI}{l_{mn}}(2\phi_{mn} + \phi_{nm}) + M_0,$$

where $\phi_{mn}, \phi_{nm} –$ rod end rotation angles, $M_0 –$ local-load bending moment affecting the rod.

For unfixed frames Bendixsen introduces a completely new second stage of the calculation, which uses states of unit displacements. Bendixsen then determines the element bias angles that are caused by displacements $w_m$ and $w_n$ of its ends, and substitutes these values in kinematic equations. The resulting node rotation angles and element bias angles are used to calculate the reactions using node equilibrium conditions.

Near 1920 Asger Ostenfeld developed Bendixsen displacement method. In Ostenfeld’s displacement method the inconsistencies inherent in Bendixsen’s approach (such as a separate determination of rotation angles of nodes and elements and two-stage determining of magnitudes of node rotation angles and element bias angles) are eliminated.
Significant progress of Ostenfeld’s displacement method [Ostenfeld, 1921] is that in the main displacement method system he introduces the node fixation both in the direction of linear displacement and the direction of rotation angles. This allows dividing the whole structure into finite elements.

"Thus, it may be expected," concludes Ostenfeld, "that this method will allow us to treat easily even very complex systems, because, in contrast to the force method, there is no need to start from nothing every time" [Ostenfeld, 1921].

8.1.3. Usage of localized functions

Discrete representation of a design scheme as a rod system was a very demonstrative method, and it was its clarity that attracted engineers. But in 1943, the distinguished mathematician Richard Courant demonstrated a fundamentally different method of problems discretization [Courant, 1943].

Solving the problem of torsion of a hollow bar with a box section, R. Courant introduced in the classical Rayleigh-Ritz method an idea of using the system functions expansion, each of functions being determined in the local area (Fig. 8.5).

This technique, which has provided, in many respects, the finite element approach, could change the history of FEM, but the Courant idea had not attracted interest of researchers, since its implementation required unprecedented amounts of computational work. Two critical elements did not exist in 1943:

- the first and most important element was the lack of a programmable computer, without which the FEM would be nothing more than a mention in a scientific book;
• also lacked pioneering publication of general form, which could attract the attention of scientists and engineers engaged in the sphere of research and development.

Fig. 8.5. The first finite-element partitioning (Courant, 1943)

Aerospace industry has provided these features only in the fifties, although the required structural mechanics apparatus (including displacement method) has become available since the late 1930’s, and Courant offer – since 1943.

8.1.4. Matrix formulation

The first publication in which the motion equations of mechanical system were presented in the matrix form, was an article by Collard and Duncan [Duncan & Collar, 1934], which introduced the concept of stiffness matrix, damping matrix and inertia matrix.

Looking back Collar describes the first attempts to use matrix algebra to formulate problems of mechanical vibrations, "Frazer had studied matrices as a branch of applied mathematics under Grace at Cambridge; and he recognized that the statement of, for example, a ternary flutter problem in terms of matrices was neat and compendious. He was, however, more concerned with formal
manipulation and transformation to other coordinates than with numerical results. On the other hand, Duncan and I were in search of numerical results for the vibration characteristics of airscrew blades; and we recognized that we could only advance by breaking the blade into, say, 10 segments and treating it as having 10 degrees of freedom. This approach also was more conveniently formulated in matrix terms, and readily expressed numerically. Then we found that if we put an approximate mode into one side of the equation, we calculated a better approximation on the other; and the matrix iteration procedure was born.”

Only after thirteen-year break a work of S. Levy [Levy, 1947] was published, which presented the matrix theory in connection with the calculation of certain aircraft designs, divided according to the force method into the bar and shearing elements. The monograph of A.F. Smirnov [Smirnov, 1947] was also published; properties of the displacement matrix, reaction matrix and influence matrix have been identified and investigated. Many elements of this monograph were pioneering.

Matrix symbols have been recognized convenient for organizing computations. Following the first publication [Levy, 1947], which was devoted to analysis of forces of the aircraft wing, there appeared the works by T. Rand [Rand, 1951], B. Langefors [Langefors, 1952] and P. Denke [Denke, 1954].

The development of this trend culminates in a series of articles by John (Ioannis) Argyris [Argyris, 1960]. The entire apparatus of classical structural mechanics was presented in detail in the matrix form, dual relationship between the force method and displacement method was traced and it is shown that both methods pass through the same sequence of steps in the process of deriving the calculation formulas. The works by Argyris and his colleagues have become the starting point for matrix reflection of known numerical methods and allowed using them with the help of computers for structures design. These works served for preparing tools for the finite element method. And the transition (Fig. 8.6) from the matrix calculation of discrete
systems to calculation of continual systems characteristic of finite element method, did not force to wait.

Describing the history of development of the finite element method, Ray W. Clough [Clough, 2004] rightly appraised Argyris’ works, collected in a book published in London [Argyris, 1955], in the following words: "In my opinion, this monograph ... certainly is the most important work ever written on the theory of structural analysis, and, when I read those articles during my sabbatical leave I immediately concluded that there was no need for me to deal with the subject of Structural Analysis Theory ..."

![Diagram](image)

Fig. 8.6. Relation of matrix displacement method (MDM) and finite element method (FEM)

However, it should be noted that the matrix transformations that so clearly and transparently describe the nature of the problem have rather primarily affected the understanding of the problem and general organization of calculations, than the computational process itself. In all nontrivial cases the matrix formulas were not used to perform calculations in the form described in textbooks of linear algebra. Almost all calculations are practically performed not over matrices but element by element over their components and in the sequence dictated by convenience of exchanges between disk and on-line memory of the computer. No one transforms the stiffness matrix
for obtaining the compliance matrix and in most cases just the
stiffness matrix formation is realized not as the product \( K = A F A^T \) (\( A \) – matrix of equilibrium conditions, \( F \) – matrix of intrinsic rigidity)
and by assembling of stiffness matrices of individual elements,
which coefficients are calculated by formulas.

8.2. Origin of the method

8.2.1. First steps

Ray Clough wrote in the report with the distinctive title *Early History of the Finite Element Method from the View Point of a Pioneer* [Clough, 2004]: "... in June 1952, I was assigned to the Structural Dynamics Unit under the supervision of Mr M. J. Turner. He was a very competent engineer with a background in applied mathematics, and several years of experience with Boeing. The job that Jon Turner had for me was the analysis of the vibration properties of a fairly large model of a ‘delta’ wing structure that had been fabricated in the Boeing shop. This problem was quite different from the analysis of a typical wing structure which could be done using standard beam theory, and I spent the summer of 1952 trying to formulate a mathematical model of the delta wing representing it as an assemblage of typical 1D beam components. The results I was able to obtain by the end of the summer were very disappointing, and I was quite discouraged when I went to say goodbye to my boss, Jon Turner. But he suggested that I come back in Summer 1953. In this new effort to evaluate the vibration properties of a delta wing model, he suggested I should formulate the mathematical model as an assemblage of 2D plate elements interconnected at their corners. With this suggestion, Jon had essentially defined the concept of the finite element method."

Here we should note words said in passing "interconnected at their corners", which should be read as "interconnected at the nodes only." This fact leads to resolving equations with respect to nodal displacements characteristic of the finite element displacement
method and unites it with the displacement method in the structural mechanics of rod systems.\textsuperscript{35}

Ray Clough obtained a solution for rectangular and triangular plate element in plane stress-strain condition in 1953, and John Turner made this report at the annual meeting of the Institute of Aeronautical Sciences in January 1954 but, for some reason, this report was not immediately submitted for publication, and only the edition of 1956 [Turner et al., 1956], almost three years after the work at Boeing was completed, became the first publication on the finite element method (Fig. 8.7)

The method obtained its name four years later in Clough’s report at the 2nd conference of the American Association of Civil Engineers (ASCE), dedicated to the use of computers [Turner, 1960].

\begin{flushright}
\textbf{JOURNAL OF THE AERONAUTICAL SCIENCES}
\end{flushright}

\textbf{Stiffness and Deflection Analysis of Complex Structures}

\textit{M. J. Turner,\textsuperscript{*} R. W. Clough,\textsuperscript{†} H. C. Martin,\textsuperscript{‡} and L. J. Topp\textsuperscript{**}}

Fig 8.7. First publication

\textsuperscript{35} Often the displacement method for rod systems calculating are also called finite element method, although this is not true. FEM is a method of approximate solution of problems described by partial differential equations. So there is just the affinity, but no more.
Turner, Clough, Martin and Topp expound briefly FEM in the form of displacement method as a six-step procedure [Turner et al., 1956]:

"(1) A complex structure must first be replaced by an equivalent idealized structure consisting of basic structural parts that are connected to each other at selected node points.

(2) Stiffness matrices must be either known or determined for each basic structural unit appearing in the idealized structure.

(3) While all other nodes are held fixed, a given node is displaced in one of the chosen coordinate directions. The forces required to do this and the reactions set up at neighboring nodes are then known from the various of individual member stiffness matrices. These forces and reactions determine one column in the overall stiffness matrix. When all components of displacement at all nodes have been considered in this manner, the complete stiffness matrix will have been developed. In the general case, this matrix will be of order $3n \times 3n$, where $n$ equals the number of nodes. The stiffness matrix so developed is singular.

(4) Desired support conditions can be imposed by striking out columns and corresponding rows, in the stiffness matrix, for which zero displacements have been specified. This reduces the order of the stiffness matrix and renders it nonsingular.

(5) For any given set of external forces at the nodes, matrix calculations applied to the stiffness matrix then yield all components of node displacement plus the external reactions.

(6) Forces in the internal members can be found by applying the appropriate force-deflection relations."

Though the pioneering publication [Turner et al., 1956] was aimed at solving problems of dynamics, it was clear that the proposed
procedure of "direct stiffness method", as it was then called by the method creators, is quite suitable for the analysis of static behavior of the structure. This became especially apparent in connection with a series of previously published works of J.Argyris [Argyris, 1954-55].

Perhaps the first study of stress-strain state performed using FEM was a calculation of a dam in Norfolk, where the thermal stresses caused a vertical crack near the section center (Fig. 8.8). Results of this study were reported in 1960 [Clough, 1960]. In this work, done in 1958-1959, he used the first world computer program that implements FEM, developed by Ed Wilson — a graduate student of Clough [Wilson, 1960].

![Fig. 8.8. The first finite-element scheme (a – object under calculation, b – FE model, c – the result)](image)

The article by Turner, Clough, Martin and Topp of 1956, which gave the start of the finite element method, along with a series of Argyris’ works, has defined the image of the first generation of FEM, which covers the period of 1950–1962. Pioneers of the method were engineers, trained in the tradition of classical mechanics, who perceived elements of the supporting structure as devices for transmitting of power. Elements discussed in [Turner et al., 1956] were specific for aircraft structures. Accordingly, thin plate elements
were considered as mechanical devices, in which, in response to nodal displacements, the nodal internal forces arise, that is in the style of displacement method. Although the classic method of forces prevailed in the 1950’s, the displacement method was fairly common in dynamics and vibration theory (note that Turner was an expert on aeroelasticity).

Further development of the finite element method confirmed the benefits of the displacement method, especially in terms of ease of algorithmic presentation of the calculation procedure. And the main role is played by the fact that in the displacement method one can quite arbitrarily choose a system of nodes (their number and position in the structure), which displacements are main unknowns.

8.2.2. Chain reaction

J. Argyris, R. Clough, G. Martin and A. Zenkevich, who were FEM pioneers and popularizers, are largely responsible for "technology transfer" from the aerospace industry to a wider range of technical applications that were made in the late 1950’s and in the 1960’s.

Ray Clough and Harold Martin, the then assistant professors of the University of California in Berkeley and the University of Washington in Seattle, respectively, worked during the summer semester 1952 and 1953 in Turner’s group, engaged in the calculation of the aircraft delta wing for Boeing company.

[Images of John Hadji Argyris, Harold Clifford Martin, and Olgierd Zienkiewicz]
The first phase of finite element method was completed by works of R. Melosh [Melosh, 1963] and B.M. Irons [Irons, 1964], [Irons, 1966]. It was shown in the paper by R. Melosh that the finite element stiffness matrix can be obtained as based on the conditions of minimizing of potential energy, it gave impetus to the transition to the variational approach in justifying FEM and to construction of various finite elements [Irons, 1964], [Fraeij de Veubeke, 1968]. And, that is no less important, the requirements were formulated in the works [Melosh, 1963], [Irons, 1966] for interpolating functions used in the construction of finite elements.

It has been proven that it is enough for convergence that a set of interpolating finite element functions satisfy the condition of completeness and to ensure deformations compatibility at inter-element borders. The requirements to polynomials issued from conditions completeness, which represent interpolating functions. But as it was found later, these conditions can be formulated in terms of mechanics – namely, the requirements of completeness are met, if, firstly, approximating functions include finite element displacements as a rigid whole and, secondly, if they make it possible to implement a homogeneous (i.e. independent of coordinates) strain state with arbitrary deformation components in the element.

8.2.3. Extension to plates and shells

Over the next decade a lot of works were published, which concerned the design and use of finite elements for two-dimensional
and three-dimensional computer models. If solution of a spatial problem of the elasticity theory has not raised new questions, and all the approaches tested for a plane problem have been preserved, such objects as plates and shells require the use of new ideas.

Several approaches have been tested for designing finite element bending plate, i.e., the transition from solving harmonic equation of a plane problem to biharmonic bending equation. Apparently, this was the first master work of V. Papenfuss from the University of Washington in Seattle [Papenfuss, 1959], and almost immediately after that there appeared the works [Adini & Clough R, 1960], [Melosh, 1961], where the so-called scheme ACM (Adin-Clough-Melosh) was introduced for the same problem; this scheme is one of the most popular one in the finite element method for the problem of bending plates. The ACM model gave the exact solution for all components of the stress state, constant within the element, but did not provide compatibility of inclination angles at inter-element borders. At the same time the Papenfuss model ensures compatibility at inter-element borders, but cannot implement the exact solution for a torque permanent in the element area.

Here, in fact, the so-called incompatible (nonconforming) element entered the finite element method [Bazeley et al., 1966]. And it may be noted that, since engineers compared with mathematicians avoid, to a less extent, the practical application of theoretically substantiated methods, no wonder that unmatched elements were first proposed by engineers. At first it caused almost no concern, the more that, though the schemes in use were incompatible, the accuracy of solutions obtained through them was quite satisfactory.

But nevertheless, when the difference between the compatible (conformal) and incompatible elements were realized, Fraeijs
de Veubeke attempted to develop a conformal finite element for a bending plate [Fraeijs de Veubeke, 1968].

Development of finite elements, modeling the behavior of shell structures, was performed in two ways:

- shell geometry itself also seemed approximate, and its smooth surface was approximated by a polyhedron, formed by plane finite elements;
- the use of curved finite elements that precisely or with sufficient degree of accuracy fit into the middle surface of the shell.

In the first case the stiffness matrix of "plane shell" element was obtained with a simple summation of stiffness matrix of plate bending element, and the element of plane problem of elasticity theory. But this approximation allowed obtaining satisfactory results only in a very large number of elements in approximating ensemble and not always ensured convergence to the exact solution, even for the case of shells of revolution. Therefore, further research had as its primary goal the construction of various types of finite elements that correctly describe the membrane geometry and provide the required solution accuracy with a limited number of elements in the ensemble.

8.2.4. Isoparametric element

It should be noted that the construction of form function (interpolating functions), which meet all necessary conditions, for the elements with curved boundaries is a very difficult task. Integrals arising in terms of element stiffness matrix and nodal vectors can no longer be calculated in a closed form. The existing difficulties can be overcome if we use the conception of isoparametric finite element proposed in 1968 by B. Irons and O. Zienkiewicz [Irons & Zienkiewicz, 1968]. In isoparametric element both the element geometry and displacement are set using the same interpolating relationships, then the transition to a numerical integration quadrature is realized. It is a combination of the two ideas that contains the nature of Irons and Zienkewicz proposition.
The work [Irons & Zienkiewicz, 1968] had a direct and significant impact on the whole range of research topics of FEM. In a short time isoparametrical elements (Fig. 8.9) gained popularity among other researchers and became a kind of standard for software developers.

![Isoparametric Elements](image)

**Fig. 8.9. Linear and quadratic isoparametric elements**

Finite element of a cylindrical shell was proposed by F. Bogner, R. L. Fox and L. Schmit [Bogner et al., 1967]. As an element of ACM, it did not provide an exact match of the strains at inter-element borders. Regarding the zero reaction to displacement of the element as a single entity, this element was endowed with this property. This combination of element properties proved sufficiently successful, and it is successfully used in the calculation practice. Other options of finite elements for calculation of membranes have not always properly allowed for the problem of hard displacement, in addition another negative feature of stiffness matrix was noticed for most of them; it was called "shear locking", when under bending of thin plates and shells, simulated by three-dimensional elements one can observe a significant increase of errors related to the display of fictitious shear deformation.

These problems become especially significant, when using curvilinear coordinates, which are introduced in order to describe better the geometry of bodies of complex shape. Then strain components depend not only on the displacement derivatives, but also on displacements and turns of the element as a whole.
To remove these shortcomings A.S. Sakharov developed a moment finite element scheme [Weinberg et al. 1971], [Sakharov, 1974a], [Sakharov, 1974b], which allowed considering the basic properties of hard shifts for isoparametric and curved finite elements of isotropic elastic bodies. The whole scheme point is that, when recording the conditions of shear deformations, those members of strain expansion in a series are rejected, which react to stiff shifts and fictitious shear deformations which appear in this case. In these conditions the exact equations of connection of deformations and displacements are substituted by approximate ones.

8.3. Search of a rigorous justification

Finite element method originated from structural mechanics and theory of elasticity, and then obtained its mathematical reasoning. A significant boost in its development FEM received in 1963 after Melosh [Melosh, 1963] proved that it can be considered as one of the variants of Rayleigh-Ritz method advanced in structural mechanics that reduces the problem to a system of linear equilibrium equations by minimizing the potential energy. Once a link of FEM and minimization procedures had been established it began to be applied to the problems described by the Laplace or Poisson equations. A little later Fraeijs de Veubeke investigated convergence and limits of the use of different finite element solutions in problems of the linear theory of elasticity [Fraeijs de Veubeke. 1965].

Following these works a number of studies on the properties of completeness and convergence of various finite element approximations was conducted. So under finite element modeling of plates Bazeley, Cheng, Irons and Zienkiewicz [Bazeley et al., 1966] proposed the following condition as a criterion of completeness: the strains would not arise during element displacement as a hard entity, and interpolation functions would admit the constant values of deformation and curvature in the element. Irons and Draper advanced similar requirements for providing convergence [Irons & Draper,
Proof of this hypothesis was later given by Oliveira [Arantes Oliveira, 1968].

The emergence of strict mathematical theory of finite elements belongs to the seventies. One can distinguish the works by I. Babushka, R. Gallagher, J. Dekla, J. Oden, G. Strang, J. Fix. Korneyev indicated the coincidence of mathematical essence of FEM and variation-difference method (VDM). FEM comparison with a number of variational methods is presented in the works of L. Rozin.

Some difficulties in justifying FEM appeared with the use of incompatible finite elements. The shape functions of these elements do not belong to energy space, and a standard method of proof of "convergence in energy" is not true here. Mathematical studies of this method were carried out, for example, in the works by Babuska and Zlamal [Babuska & Zlamal, 1973], Ciarlet [Ciarlet, 1974], G. Strang and J. Fix [Strang & Fix, 1973]. It turned out that convergence depends essentially on the certain individual features of schemes.

Quite general method of convergence of incompatible finite element systems and the way of constructing converging incompatible elements were indicated only in 1981 by I.D. Evzerov [Evzerov, 1981].
8.4. Other variants of FEM

8.4.1. Ritz method and Galerkin’s method

The functional, which is a potential (Lagrange functional) or complementary (Castigliano functional) system energy is used in mechanics of deformable bodies. If we substitute approximating expressions of desired functions into the functional and apply the extreme principles to it, we obtain a system of algebraic equations; the values of nodal unknowns will be its solution.

The sphere of the FEM use has considerably expanded when Oden [Oden, 1969] and Szabo and Lee [Szabo & Lee, 1969] found that the equations that define elements in the problems can be easily obtained using the method of weighted residuals such as Galerkin’s method or least squares method.

Indeed, using variational principles is not necessary for obtaining resolving equations (in particular, the potential function may not exist). Then a discrepancy is involved in consideration – a deviation of the approximative solution from exact solution of differential equations for this problem. To get the "best" solution, it is necessary to minimize a certain residual integral over the calculation area. To improve efficiency the so called weight function in introduced in the integrand near the residual. The choice of minimization diagram and weight functions determines various variants of the method of residuals. Galerkin’s method is the most commonly used one that, the energy functional being available, leads to the same equations as the variational approach.

Justification of FEM using the Galerkin method has played a significant part in the theoretical justification of the method, since allowed its use in solving many types of differential equations [Demkowicz & Oden, 1986]. Thus, the finite element method has become a general method of numerical solution of differential equations or systems of differential equations.
8.4.2. The use of other functionals

In problems with peculiarities (such as the consideration of weak compressibility, calculation of plates and shells based on three-dimensional finite elements) serious difficulties appear, when using traditional schemes of FEM in the form of the displacement method [Irons, 1970]. Other variational principles are used to overcome the above difficulties: Castigliano method of forces, Hellinger-Reissner, Hu-Washizu mixed method.

FEM in the form of the force method was originally used by de Vebeke [Fraejs de Veubeke, 1965], but this variant of the method has not been significantly developed because of the difficulties in the stress state approximation. Mixed FEM schemes were more widely used. The first work on the mixed method was published by L. Herrmann in 1965. [Herrmann, 1965] and was associated with analysis of a bending plate. A set of mixed finite elements for plates and shells was subsequently developed (Prato [Prato, 1968], J. Connor [Connor & Will, 1971], A. Poceski [Poceski, 1975, 1979], F. Brezzi [Brezzi et al. 1985, 1987]). In these and further works on the mixed method the theory was based on the application of variational principles and use of the functionals of full potential and complementary energy. The displacement and force fields in FE volume were provided as independent of each other. The abstract mathematical analysis of these methods was first performed in the following works [Aubin & Burchard, 1970] and [Babuska & Zlamal, 1973].

Having positive features, such elements also have a number of shortcomings such as the increase in the order of resolving system of equations compared with FEM method in the form of displacement method, violation of positive determinability of the system matrix. Therefore, the development of hybrid FEM scheme in the form of the displacement method based on Lagrange variational principle is more promising for the problems with above peculiarities. The first work dedicated to hybrid finite elements, was published by T. Pian and P. Tong [Pian & Tong 1969]. These authors have constructed a set of
functionals, the fields of some functions in them (such as displacement functions) vary independently throughout the domain of definition of the problem solution, while other functions (e.g., stress functions) vary only at some surfaces (lines) that divide this domain into sub-domains that do not overlap. The finite element schemes corresponding to these functionals are called by the authors the hybrid FEM schemes. Interestingly, the same idea of independent variation of different functions (deflections and rotations) in the domain and at inter-element boundaries of finite elements is used in the work of Japanese scientists Kikuchi and Ando [Kikuchi & Ando 1972], that allowed them, using simple means, to overcome certain difficulties of FEM in the theory of bending of thin plates associated with the requirement of continuity of the first derivative of the deflection function.

It should be noted that the auxiliary unknowns, used in mixed and hybrid versions of FEM, are usually associated with the derivatives of the unknown displacement and have a certain physical sense, but sometimes their calculation is even of more practical interest than the data about basic unknowns.

But mixed functionals have two major drawbacks:

• lack of extreme properties (stationary point is not a point of extremum for reissnerian) generates difficulties of computational nature;

• order of a mixed equations system is the sum of orders due to a displacement approximation and stress approximation.

These shortcomings make us to cautiously approach the mixed functional.

With the development of FEM and increase of the number and quality of its software implementations the method shortcomings also become apparent. The main drawback of FEM is realized in a form of displacement method – that is, paradoxically, the continuation of its main advantage.

Indeed, the most important feature of variational formulation of the problem, using the functional of full potential energy of the
system, is the reduction of requirements on the smoothness of solutions. This extension of the domain of the solution definition on the one hand helps construct space of displacement functions admissible for comparison, on the other hand, it is too "weak" for stresses, because the operation of differentiation necessary in the transition from displacement to stress can disrupt the continuity of stress state component.

In order not to give up benefits, researchers began looking for ways to get rid of the shortcomings or at least to smooth their negative influence.

In many cases, the engineer has a priori information about the smoothness of stresses in exact solution of the problem and, if this information is contrary to observation (expected) smoothness, there appears a desire to "enoble" the solution.

Thus, the problem is to construct a smoothing operator to project a discontinuous solution to some surface, which smoothness is consistent with the available prior information. For a time hopes were pinned on the Barlow idea, following which stresses are searched in some “optimal” points other than the nodal ones, where the stress error is warrantly minimum. This idea was not widely used because of the difficulties, which appeared in a search of optimal points, moreover, users are usually interested in stress in the nodal points of the finite element mesh.

Interesting idea of obtaining nodal stress (force) values in FEM belongs to A.V. Vovkushevsky [Vovkushevsky, 1976]. According to his proposal, along with the stress node he considers a "star" of its elements, i.e. the domain consisting of finite elements adjacent to this node. The very node is called the star center. A new displacement approximation is introduced within the star; it is not related, generally speaking, to the approximation that is used in determining the nodal values of displacements, but it is such that new approximation in each node of the star satisfies the displacements given by the finite-element solution. To this end, a system of
equations is composed that is overdetermined, so its solution is to be searched, for example, by the method of least squares.

The so-called Oden’s method of combined approximations [Oden & Reddy 1973] is very popular in the world technical literature. Unfortunately, its variational interpretations (Oden does not connect his method with any variational formulation of a problem) shows that the resolving equations follow from stationarity conditions of a functional that has no explicit physical sense. This shortcoming was eliminated in the work by V.I. Slivker [Slivker, 1982], where he proposed a scheme of FEM characterized by the following features:

- independent approximations are taken for the force and kinematic fields;
- two Ritz systems of equations are built, each of them has positively defined matrix;
- boundary conditions (both static and kinematic) are satisfied exactly.

Unlike the traditional approach, based on fulfilling conditions of one functional stationarity, two functionals are connected with differential problem statement in the above method, their minimum conditions on the finite-dimensional subspaces give the desired systems of equations.

When analyzing nonlinear behavior of shell structures, one of the problems is that isoparametric degenerate elements are most widely available in commercial software packages,  that consider a shell as a set of plane membrane elements, some classes of problems show low productivity. Geometrically precise elements are the alternative to degenerate elements. In geometrically precise elements the membrane reference surface is described by analytically given functions, including splines, and the term "geometrically precise" means that the coefficients of the first and second fundamental forms of the reference surface and Christoffel symbols are calculated accurately at each node. Such finite elements of the shell prone to
large displacements and rotations, have been recently built in the work [Kulikov, Plotnikova, 2011].

8.4.3. Discrete-continual (semi analytical) FEM

A transition to a system of ordinary differential equations is possible instead of transition from a system of differential equations to algebraic equations which is characteristic of the finite element method. For the first time this idea was probably suggested by Academician L.V. Kantorovich [Kantorovich, 1934], and that was a development of the Ritz method. This type of approach has been used by V.Z.Vlasov in the problems of structural mechanics since 1931.

His approach has ideological affinity with the Bubnov-Galerkin method. The difference is only in the form of setting the approximate expression for the unknown function of several variables: in the Bubnov-Galerkin method unknown constants are taken as coefficients of the coordinate functions; a system of linear equations is composed to determine these constants, while in the V.Z.Vlasov method unknown functions on one independent variable play the role of coefficients. These functions are determined by solving a system of ordinary differential equations.

From a viewpoint of mechanics the design model of the structure corresponds to this method; this model is a system which has a finite number of degrees of freedom in one coordinate direction and infinitely large number of the degrees of freedom in the other coordinate. V.Z.Vlasov has called such systems the discrete-continual ones, and the method itself is often called Vlasov’s discrete-continual method [Shaposhnikov, 2009].

The method of lines (differential-difference method) is similar to Vlasov and Kantorovich methods – it is a very effective method to reduce the dimensionality of initial boundary problems. In this method the derivatives of one of independent variables in two-dimensional problems and the two independent variables in three-dimensional problems are replaced with difference expressions.

442
The use of this procedure provides replacement of the boundary value problem for partial differential equations by the boundary value problems for ordinary differential equations. This is the main feature of the method of lines. That was proposed before the advent of computers and efficient highly accurate algorithms for solving boundary value problems for the systems of ordinary differential equations.

The method of lines was first adapted to direct solving the static problems of elasticity theory (two-dimensional and three-dimensional) by M.G. Slobodyansky [Slobodyansky, 1939]. The variant of direct method for solving spatial problems of the elasticity theory was proposed in the work of L.P. Vinokurov [Vinokurov, 1951, 1956] in structural mechanics. This method was first used by Ya.M. Grigorenko and his students [Vlaikov et al., 2001] to solve the problems of shell theory.

The method generalization was rather natural, when in solving the two-dimensional problem the difference procedure of one coordinate (or two coordinates for three-dimensional case) is replaced by the finite element approximation of solution. This finite strips method was proposed in 1968 by I.Cheung [Cheung, 1968, 1996].

The finite strips method combines the idea of Kantorovich-Vlasov analytical method and technique of the finite element method. The so-called discrete-continual model of object is used. For example, the three-dimensional structure is broken with longitudinal sections (with linear nodes or so-called nodal lines) and thus it is discretized in only one transverse direction. The result is an ensemble of rectilinear or curvilinear finite elements (in two-dimensional case – strips). The ordinary finite element partition is used in other (transverse) directions.

443
Kinematic unknowns can be chosen as nodal unknowns: displacement, rotation angles and so on. This is the so-called "stiffness" variant. The stiffness variant has three main submissions: semi analytical method, analytical and numerical methods.

Trigonometric series (especially in cases of hinge fixing of limit cross-sections), beam functions, polynomial functions jointly with trigonometric functions and other analytical functions are used in semi analytical method as a basis in the longitudinal (analytical) direction. The shape functions of different types are the basis of cross (discrete) direction.

Exact solutions of corresponding differential equations, modeling the problem, are used in the analytical method as the basis in transverse direction. On the whole, the analytical method is not widely used. But since it is the most accurate one, it is used for testing other methods.

The splines which fulfill necessary conditions of continuity or rupture are chosen in the case of numerical variant for the basis under the approximation of longitudinal displacement. The shape functions are used in the transverse direction, as well as in the case of semi
analytical variant. A numerical variant of finite strips method is used much more rarely than semi analytical one.

The main unknowns (further the basic nodal unknowns) are considered with respect to nodal lines. On these lines, the main unknowns are replaced by the continuous method along the longitudinal coordinate and in a classic case they are presented as a sum of some basic functions that satisfy the boundary conditions. Trigonometric functions are used as basic ones, for example, if a nodal line is a circle, or Michlin’s polynomials [Bazhenov et al., 1993] for the nodal lines of the other type.

This approach was further successfully developed to solve a wide range of problems of linear and nonlinear deformation of space bodies of canonical form – inhomogeneous circular and prismatic bodies. So in the works [Bazhenov et al., 2005] and [Bazhenov et al., 2014] the solutions of the problems of thermoviscoelastic deformation with allowance for material damage were obtained, and the problems of stationary and nonstationary vibrations under loads of different levels of intensity and duration in time was considered in the work [Bazhenov et al., 2012].

Standard semi analytical approaches do not cope with the allowance for loads, being distributed in small areas along the nodal lines, in particular – concentrated loads. The boundary conditions are no less critical in the same way: some special kind of conditions that has no place in the general case is required for their adequate consideration. The accuracy and convergence of solutions, obtained by such methods, are often highly dependent on the kind of chosen basis functions for unknowns determination, and on the number of accounted terms of the series. At the same time, the convergence in the zones of edge effects, concentrated factors, concentrations of stresses and strains (i.e. in the most crucial zones) is very slow and poorly depends on the number of terms of the Fourier series, that is partly explained by the Gibbs effect well-known in the theory of series.
To overcome these difficulties a variant of discrete-analytical method was recently proposed, which uses a direct solution of a multi-point boundary value problem for the systems of ordinary differential equations. As part of this approach O.B. Zolotov and P.A. Akimov [Zolotov and Akimov, 2004] implemented a stable algorithm of analytical solution under any number of unknowns in the form that is correct for calculations.

8.4.4. Superelement approach

The finite element method has quickly proved itself to be a powerful tool for analyzing complex engineering structures. Now one can solve very large problems under availability of computers with appropriate capacity, and that was the circumstance, which established the bounds of the possibility. Hence appeared the need for FEM modifications focused on solving high complexity problems. One of these modifications was the superelement method.

The idea of an independent calculation of individual parts of the system (subsystems) and further consideration of their interaction is assumed as a basis of superelement method (method of substructures). It is suggested that all the nodes of the system are divided into two sets – external and internal nodes, and unknown displacements of the system are considered as the sum of two components. The first component – the displacements caused by external forces in restraint of borders in the subsystems. Displacements of each subsystem are determined from equations involving unknowns, associated only with this subsystem. The second component – the displacements caused by displacements of subsystem borders that excludes internal nodes.
The system partitioning into substructures was introduced by experts in aerospace industry in the early 1960’s and then was considered as the method of "solving problems by parts" [Crown, 1972]. To solve the problems of the elasticity theory the method of substructures was first used by A.N. Przemieniecki [Przemieniecki, 1963].

The book [Przemieniecki, 1968] contains a full bibliography of earlier works, most of which had the status of largely inaccessible internal communications of companies or laboratories. Thus, the actual history is difficult for tracing, that is why the Przheminitskii priority is rather conventional. Significant development of superelement approach was obtained in the work of Meissner [Meissner, 1968], who formalized the method and generalized it on several levels of superelements nested one in one.
The simplest superelements appeared simultaneously in numerous early programs of FEM (e.g., rectangular element composed of triangles), but a superelement of general purpose was proposed in the late 1960’s by a group SESAM in Det norske Veritas company [Egeland & Araldsen 1974]. When calculating the structure by the superelement method first the problem dimension is decreased by eliminating (reduction) the internal degrees of freedom, when displacements of the internal nodes of the structure are expressed through displacements of external nodes. For identical substructures the stiffness matrix is reduced only once, that significantly decreases the calculation time.

Elimination of internal unknowns in the matrix form of static compaction was proposed in the work of Guyan [Guyan, 1965] as a method of removal of "non-essential" degrees of freedom in dynamic analysis. Only later such an approach as well as incomplete factorization method became the basis of numerous superelements algorithms. They differ one from another mainly by the system partition into subsystems. Here the approaches are used based on repetition of parts of the structural complex or functional ones, when superelements are connected with parts of the structure (modules) that have the same functional purpose [Postnov and Taranukha, 1990].

Technology of superelementis has found application in the so-called local-global analysis proposed in the work [Mote, 1971] and oriented to studying behavior of the structure with marked disturbance of the stress-strain state in the local area. The idea of the approach is that we first solve the global problem, which ignores local disturbance, and then a solution with allowance for disturbance in a distinguished subdomain (kind of superelements) is sought.

Using the domain partitioning into subdomains caused a new wave of interest in recent years in connection with the problem of paralleling computation process for multicore computers [Kopysov, 1999].
8.5. Software implementations

8.5.1. Choosing of the method

Between the late 1950’s and early 1960’s, when there appeared a possibility of access to computers in design organizations and design offices, the classic force method dominated, though there also appeared the adherents of the displacement method. Engineers brought up in the 1940’s and 1950’s, were often the adherents of transition to computer calculations, and no wonder that considerable effort has been devoted to programming the force method [Robinson, 1966], [Reznikov, 1971].

The author of the book [Reznikov, 1971] R.A.Reznikov, who was an adherent of using the force method, gave epigraph to the chapter, which was devoted to this method: "Good melody can be played with the old violin." But in technique, unlike art, "the old violin" will never win the contest without a substantial modernization, and eventually, the displacement method became dominant in software implementation for general purpose. Programs based on the force method remained only in some object-oriented elaborations.

There is the typical history. In 1965, NASA assignment to develop finite element program NASTRAN contained a requirement to develop simultaneously the versions of the displacement method and the force method [MacNeal, 1988]. As expected, each version was to have identical simulation capabilities and possibilities of solving problems, including the problems of dynamics and large strains of the objects. Ultimately, development of the version of the force method was abolished in 1969.

And describing the result of competition of these methods, you can bring Einstein's response to question about the reaction of the old school to the new physics, "We did not convince them; we have outlived them.”

Note that saying here about the displacements method, we always had in mind its modern interpretation focused on the software
support. In those days, when the basic computing tools of an engineer were a pencil, paper and a slide rule, the calculations of frame structures, for example, usually silently implied that the rod frames were incompressible. In manual calculations this assumption reduced the amount of computational work by reducing the number of degrees of freedom of the rod system nodes. The order of the stiffness matrix decreased, and benefit from that surpassed all the drawbacks associated with the hypothesis of incompressibility.

But what was good yesterday is not the best today. In particular, some things useful for manual calculation, are not necessarily useful for computerized calculations. Consideration of rods compliance in their longitudinal direction actually leads to the increased number of the degrees of freedom of nodes, but simplifies the algorithms of forming the overall stiffness matrix system, and as for software developers, the latter circumstance is more important and significant.

8.5.2. Formation of software architecture

Development of computer programs that implement FEM started simultaneously with the appearance of this method. Ed Wilson, a graduate student of Turner, was involved in this work. In July 1962, John Turner wrote [Turner et al., 1964]:

"The report, presented in 1959 at a meeting of the staff of Designs and Materials Division AGARD\(^{36}\) in Aachen (Germany), described the essential features of the system for the numerical analysis of structures using the direct stiffness method. A characteristic feature of this particular version of the displacement method is the procedure of composing, resulting in filling a stiffness matrix for a complex system with direct addition of matrices associated with the system elements".

\(^{36}\) AGARD (Advisory Group for Aerospace Research and Development) - Advisory Group for Aerospace Research – Science and Technology Agency, which existed in the structure of NATO from 1952 to 1996.
This was one of the key elements for all further developments and, incidentally, the first of many departures from the direct use of matrix operations. Numerous other departures of this kind occurred later, when the question arose of the work with external memory.

The first program created by E. Wilson had not been the development for industrial use. It worked in the author’s hands, and such were other software developments of the late 1950’s and early 1960’s. During the first years each researcher developed his own computer program or changed the program of other researcher to analyze specific types of structures. These programs were not often registered and could not be used by anyone else except the developer. For these reasons, Ed Wilson in 1969 began development of the public use program SAP (Structural Analysis Program) for the purpose of static and dynamic analysis [Wilson, 1970].

SAP program used the existing technology of that time. Each node could have from zero to six degrees of freedom of displacements. Within SAP the indexes were created for nodes that allowed each node to have a variety of unknown displacements. The equilibrium equations were formed in the process of building a stiffness matrix only for the unknown displacements. Thus, the program was as effective as specialized target programs that had the incomplete number of displacements in the node.

In 1973, Klaus-Jürgen Bathe updated dynamic branches of the program and created a variant SAP IV [Bathe et al., 1993]. At the time of its completion SAP IV was one of the fastest programs and had the greatest potentialities in the world among similar computer codes.

But the first commercial software package for general purpose, made in 1969, was the package ASKA (Automatic system for kinematic analysis) [Argyris, 1969]. It was developed in Germany at the Institute of Statics and Dynamics of Aerospace Structures under the direction of J. Argyris.

ASKA program was focused on a certain type of machine, but the rapid development of computer technology have led to the need
of creating the program in a language understandable to any machine. Features of algorithmic language FORTRAN have determined its widespread use for programming, when solving problems by the finite element method.

The NASA program was an attempt to create a flexible program for extensive research of problems of the US aerospace industry. Programs FESS (Finite Element Solution Swansea) and FINESSE created in Swansea were more focused on the efficient solving of small and medium engineering problems of structural mechanics, such as the calculation of bridges, dams, nuclear reactors. Main attention was paid to creating a simple system that can easily adapt to any specific problems.

This ideology was the basis of domestic developments, such as software system LIRA or complex SCAD Office. These developments targeted at mass use of computer technology, were improved with the growth of technical potential (but not only at this point), and now, when they can solve the problems with several millions of unknown displacement, it is hard to consider them the programs for solutions of small and mean problems. At the same the mass user-orientation affected simplification of input of initial data, development of controls and visualization of results.

8.5.3. Search of solvers

Among the methods for solving the systems of linear algebraic equations with sparse matrices both direct methods and iterative methods are used in FEM programs. The iterative ones are usually used when the equation system matrix is positively determined.

The earliest programs were used by iterative solvers as sizes of the problems to be solved exceeded potentialities of first computers. First generation computer had enough speed, but little of memory. E. Wilson writes "... my first computer Univac had 1,000 of 45-bit words, and IBM-701 – 2048 of 36-bit words. It is clear that solving the complete system of 100 equations was then a serious challenge" [Wilson, 1993].
Later programs were specified by intensive using the direct methods, including the prevailing tape method, and then the profile method [George & Liu, 1981], based on the Gaussian elimination. In the 90's the methods that allow for the sparse matrix structure became more popular. Owing to effective ordering algorithms that significantly reduce filling during the factorization process, developers managed to significantly reduce the size of factorized matrix and calculation time. However, these programs are effective only when the whole matrix is placed in the on-line memory, that is for relatively small problems.

This forced the software developers to apply the frontal method, the technology proposed by Bruce Irons [Irons, 1970], in which the assembling and elimination of fully assembled equations is conducted in parallel. The system stiffness matrix in an explicit form is not assembled, but instead, element by element are added. As soon as the next node is assembled, that is all the elements adjacent to it are included in the ensemble, all unknowns belonging to the node and equations associated with them are eliminated. Therewith the addition of the next elements does not bring any changes to them. As a result the Gaussian elimination is made in a dense (frontal) matrix in respect of small dimensions; the matrix consists of two parts, one of which is fully assembled. The fully assembled equations are immediately eliminated, and the relevant part of the matrix is recorded to disk. Then another finite element is added and the assembled equations are eliminated.

Most crucial moment in using direct methods is the ordering of equations aimed at reducing fillings. Due to the fact that the optimal solution of the problem is very expensive they use in practice heuristic algorithms, minimum degree algorithm, nested dissection method, parallel section method and multiple sectioning method being the most common among them [George & Liu, 1981].

When using these methods of ordering in combination with technique of the frontal method, as a rule, there occur several fronts, because under a given sequence of elimination of nodes of the finite
element model the incompatible groups of elements are formed during the considerable part of assembling-elimination process. The appropriate approach, being efficient both for parallel computations and for ordinary computers, is known as multifrontal method. Today it is one of the most efficient and perhaps the only method of factorization of large sparse matrices in industrial software complexes [Fialko 2009].

Fig. 8.12. Stages of Irons method (assembled part is darkened):
- front nodes, • – nodes with solved equations

The block multifrontal method developed by S.Y.Fialko is distinguished by complete automation of the original structure division into substructures, based on the natural union of equations in the groups associated with nodes of the design model. In this sense, the method is a generalization of superelement method and differs from the latter by automatic structure partition into superelements.

The iterative methods for solving systems of resolving FEM equations have not also gone unnoticed. They are usually used in studying very large problems of structural mechanics. This especially concerns the problems on calculation of three-dimensional objects with a fairly dense finite element partitioning.

The method of conjugate gradients with precondition is the most used one among the perfected iterative methods. Precondition – a powerful way to accelerate the convergence of iterative methods.
That is to construct positively determined (usually symmetric) matrix, which is multiplied by the given coefficient matrix, resulting in a transition to transformation matrix, which conditioning number should be less than in the corresponding output matrix.

It is known that for most problems of static calculation of structures the smooth components make a significant contribution to the solution; finite element model with a small number of nodes is required for their satisfactory approximation. Multilevel (multimesh) iterative methods are used for allowance for this peculiarity. A detailed finite element model with high-degree mesh partitioning is required only for approximations of the part of solution that is rapidly changing. A certain necessary degree of discretization of finite element design model corresponds to each “smoothness range” for solution components. In this regard, it is natural to develop the iterative process of solving the problem on a sequence of meshes, the smallest of which coincides with the most detailed partitioning of finite element structure, and the rest are auxiliary, the problem solution at the largest mesh (coarse level model) should be performed by direct method.

The multimesh iterative solving method was used by A.Vovkushevsky [Vovkushevsky, 1976], who demonstrated significantly better convergence than traditional iterative methods. A very effective variant of the multimesh semi-iterative method to calculate the three-dimensional massive structures was proposed by O.B.Zolotov, M.V.Belyi and V.E.Bulgakov [Zolotov et al., 1985].

8.5.4. Step-by-step procedure

To solve nonlinear problems the creators of FEM in 1956 recommended a step-by-step procedure, which was quite natural for considered dynamics problem, when this procedure is used to solve the Cauchy problem.

In static problems the idea of continuation of parameter solution for computing purposes was first introduced by M. Lahaye [Lahaye, 1934]. He introduced a parameter $p$ in transcendental equation so that
the solution could be easily obtained at $p = 0$, and when $p = 1$ the equation became the output one. As parameter $p$ increased, M. Lahaye proposed to build a solution for each $p_i$ value by Newton-Raphson method, using the solution for the previous step $p_{i-1}$ as the initial approximation.

Other formulation of the method belongs to D.F. Davidenko [Davidenko, 1953], who used it (under the name "method of variation of parameter") in a wide class of the problems of applied mathematics that need solving the nonlinear operator equations of the form $F(x,p)=0$. He was probably the first to realize the process of parameter continuation as motion and applied the apparatus of differential equations to this process. These equations were linear with respect to derivatives $dx/dp$ and were solved with the help of any of the known numerical methods (Euler, Runge-Kutta, Adams, etc.).

The Euler method is the most commonly used one; it does not provide the compensation of calculation errors caused by linearization of nonlinear equations at every step. Therefore, the required accuracy can be achieved by reduction of the load increment value. There are also variants of implicit schemes of the Cauchy problem parameter integration using different ways to improve the convergence of iterative processes such as the Newton-Raphson method.

Independent of the works by Davidenko, following the idea of V.Z. Vlasov, V.V.Petrov has formulated the well-known method of successive loads for solving the problems on geometrically nonlinear deformation of shells [Petrov, 1959]. Based on the assumption of smallness of deflection and stress at small load increments, V.V.Petrov has formulated a recurrent sequence of linear boundary value problems for determining these increments.

The choose of the continuation parameter is a key factor, ensuring the possibility of solution continuation in a neighborhood of singular points. In the articles [Feodosev, 1963] and independently [Vorovych, Zypalova, 1965] it was proposed to implement the solution continuation along the curve of equilibrium states in the
space of \( m + l \) coordinates and load parameter within the continuum-based model. It was proposed to use the curve length increment as continuation parameter.

In the problems of discrete nonlinear structural mechanics the method of parameter continuation, which uses the arc length of the curve of equilibrium states as a parameter, was first proposed by Ricks [Ricks, 1972] and Wempner [Wempner, 1971]. The following modifications of this method were made by Crisfield [Crisfield, 1980]. He proposed a modification of the method in which the arch chord length of the curve holds constant. Thus, the problem solution is found in the intersection of the true curve of equilibrium states and the sphere with the center in the previous calculated plotted point and radius equal to the set length of the chord.

V.I.Shalashylin and E.B.Kuznetsov have obtained a mathematically rigorous proof that such a choose of the parameter of solution continuation in the form of arc length of the curve representing the solution of the nonlinear equation, is the best [Shalashylyn, Kuznetsov, 1999].
References


Aleksandrov, A.V., Leshchenkov, B.Ya., Shaposhnikov, N.N. and Smirnov, V.A. (1976), Methody raschota sterezhevykh system, plastin i obolochek s isplozovaniem ECVM [Methods of analysis of bar systems, plates, and shells using EDC], Stroyizdat, Moscow, USSR.


Vinokurov, L.P. (1956), Pryamye metody resheniya prostranstvennykh i kontaktnykh zadach dlya massivov i fundamentov [Direct methods of solution of space and contact problems for massifs and foundations], KhGU, Kharkov, USSR.
Vovkushevsky, A.V. and Shoikhet, B.A. (1981), Raschot massivnykh gidroteknicheskikh sooruzhenii s uchotom raskrytiia shvor [Analysis of massive water development works with allowance for joint opening], Energiya, Moscow, USSR.
Yevzerov, I.D. (1981),”Convergence of FEM in case of basis functions not belonging to energy space of the basis functions”, Vychislenniya s razrezhennymi matritsami, VCSO AN SSSR, Novosibirsk, USSR.


Korneyev, V.G. (1977), Skhemy metoda konechnyk elementov vysokikh poryadkov tochnosti [Schemes of the method of finite elements of high orders of accuracy], Izdatelstvo Leningradskogo universiteta, Leningrad, USSR.

Kron, G. (1972), Issledovanie slozhnykh system po chastym (diakoptika) [Study of complex systems by parts (diacoptics)], Nauka, Moscow, USSR.

Maslennikov, A.M. (1966), Priblizhennoe reshenie ploskoi zadachi teorii uprugosti metodom peremeshcheniy [Approximate solution of a plane problem of elasticity theory by the displacement method], EDC in structural mechanics, Sudpromgiz, Leningrad, USSR.


Oden, J. (1976), Konechnye elementy v nelineinoy mekanike sploshnykh sred [Finite elements in nonlinear mechanics of continuum], Mir, Moscow, USSR.


Petrov V.V. (1975), Metod posledovatelnykh nagruzheniy v nelineinoy teorii plastin i obolochek [Method of successive loadings in nonlinear theory of plates and shells], Izdatelstvo Saratovskogo universiteta, Saratov, USSR.

Postnov, V.A. and Taranukha, N.A. (1990), Metod modul-elementov v raschotakh sudovykh konstruktsiy [Method of module-elements in calculations of boat structures], Sudstroenie, Leningrad, USSR.

Raschot stroitelnykh konstruktsiy s primenieniem elektronnykh mashin [Analysis of building structures with the use of electronic computers], Stroyizdat, Moscow, USSR, 1967.

Reznikov, P.A. (1971), Reshenie zadach stroitelnoy mekhaniki na EVM [Solution of the problems of structural mechanics by EDC], Stroyizdat, Moscow, USSR.

Rzhanitsin, A.R. (1956), Predstavlenie sploshnogo izotropnogo uprugogo tela v vide sharnirno-stershnevoi sistemy. Issledovaniya po stroitelnoy mekhanike i
teorii plasticnosti [Presentation of a continuous elastic body in a form of hinge-bar system]. Studies in structural mechanics and elasticity theory, Gosstroyizdat, Moscow, USSR.

Rozin, L.A. (1977), Raschet gidrotekhnicheskikh sooruzheniy na ECVM. Metod konechnyk elementov [Design of water development works by EDC. Metod of finite elements], Energiya, Leningrad, USSR.

Rozin, L.A. (1977), Metod konechnyk elementov v promenii k uprugim sistemam [Finite element metod as applied to elastic systems], Stroyizdat, Moscow, USSR.


Slivker, V.I. (1982), Stroitelnaya mekhanika. Variatsionnyie osnovy [Structural mechanics. Variational principles], Izdatelstvo ACB, Moscow USSR.


Smirnov, A.F. (1947), Staticheskaya i dinamicheshkaya ustoichivost sooruzheniy [Statical and dynamical stability of structures], Transzheldorizdat, Moscow, USSR.


Fialko, S.Yu. (2009), Pryamyie metody resheniya system lineynykh uravneniy v sovremennykh MKE-kompleksakh [Direct methods of solution of the systems of linear equations in modern FEM software complexes], Izdatelstvo SCAD SOFT, Moscow, Russia.

Shalashilin, V.I. and Kuznetsov, E.B. (1999), Metod prodozheniya resheniya za parametrom i naiuchshaya parametrizeatsiya v prikladnoi matematike i mekhanike [Method of solution parameter continuation and the best parametrization in applied mathematics and mechanics], Editorial URSS, Moscow, Russia.


Brezzi, F. (1991), Mixed and hybrid finite element methods, Springer-Verlag, New York, USA.
Clough, R.W. and Wilson, E.L. (1960), Early finite element research at Berkelay, Fifth U.S. National Conference on Computational Mechanics, Aug. 4-6, 1999
Csonka, P. (1962), Beitrag zur Berechnung waagerecht belasteter Stockwerkräumen, Bautechnik, no. 7

463
Demkowicz, L. and Oden, J.T. (1986), An adaptive characteristic Petrov-Galerkin
finite element method for convection-dominated linear and nonlinear parabolic
problems in two space variables, *Computer Methods in Applied Mechanics and
Duncan, W.J. and Collar, A.R. (1934), A method for the solution of oscillations
Euler, L. (1789), De motu vibratorio tympanorum, *Novi Comm. Acad/ Petrop.*,,
T. X.
Fraeijs de Veubeke, B. (1968), A conforming finite element for plate bending,
Fraeijs de Veubeke, B. and Sander, G. (1968), An equilibrium model for plate
447-468.
Fraeijs de Veubeke, B. (1965), Displacement and equilibrium models in the
George, A. and Liu, J. (1981), Computer solution of large sparse positive definite
systems, Englewood Cliffs: Prentice-Hall, NJ, USA.
3, no. 2, p. 380
Str. Mech.*, Wright Patterson AFB, Ohio, USA.
Hrennikoff, A. (1941), Solution of problems of elasticity by the framework
Limited, Chichester, USA.
Irons, B.M. (1970), A frontal solution scheme for finite element analysis,
Irons, B.M. (1966), Engineering application of numerical integration in stiffness
Irons, B.M. (1964), Comments on 'Matrices for the direct stiffness method' by R.
Irons, B.M. and Zienkiewicz, O. C. (1968), The isoparametric finite element
system – A new concept in finite element analysis, *Proc. Conf. Recent


MacNeal, R.H. (1988), The MacNeal Schwendler Corporation: The first twenty years, Gardner Litograph, Buena Park, CA, USA.


Prato, C. (1968), A mixed finite element method for thin shell analysis, Ph. D. Th. Dept. Civil Eng. MIT.
Ricks, E. (1972), The application of Newton's method to the problem of elastic stability, Trans. ASME, E39, no. 4, pp. 1060-1065.
Robinson, J. (1966), Structural matrix analysis for the engineer, John Wiley & Sons, New York, USA.
Wilson, E.L. (1970), SAP-A general structural analysis program, UCB/SESM Report No. 70/21, University of California, Berkeley, USA.
Essay 9

STAGES OF DEVELOPMENT OF THE PROBLEMS OF SYNTHESIS IN STRUCTURAL THEORY
Le mieux est l’ennemi du bien (Better is the enemy of good)

Voltair

A real engineer must trust his eye more than any formula; he should remember the words of naturalist and philosopher Thomas Huxley: "Mathematics may be compared to a mill of exquisite workmanship, which grinds your stuff to any degree of fineness; but nevertheless, what you get out depends on what you put in.”

A.N. Krylov
Introduction

As Eduardo Torroja, the famous Spanish architect and engineer, [Torroja, 1967] mentioned "The best structure is the one which reliability is ensured mainly by its form, not by the strength of its material." When designing building structures, an engineer faces the problem of compliance with such requirements as strength, stiffness, stability, durability, economic efficiency, serviceability, terms of design and construction, the use of certain resources and materials. All these requirements are quite controversial, since the project optimization is the main goal of every engineer, who wants to create a certain element, design or structure that meet certain criteria.

The problems of optimal design have long attracted great attention, the significant number of works is devoted to them and the history of optimum design goes back nearly four centuries and comes from the work by Galileo Galilei [Galilei 1638], a core work of structural mechanics. This essay is not aimed at giving a detailed history of the problem with all its branches in various fields and applications. It seems that self-interest may be the history of emergence and development of the basic ideas that can be illustrated by focusing mainly on bar systems that have long been the only object of research in structural mechanics.

Moreover, even among bar systems we can identify trusses that have the peculiarity that each element of the system is characterized by only one parameter, namely the cross-sectional area (a similar situation is realized in the case of three-layer systems that are exposed to bending, if the filler thickness is set and the identical top and bottom thicknesses of bearing plates are to be determined) [Prager, Taylor, 1968]). This feature considerably simplifies the analysis, and the basic techniques of solving optimization problems were perfected just on these trusses. Such approach allows describing the history of development in a "pure" form, i.e. without the
influence of numerous additional factors, being important for practical reasons but greatly complicating the analysis.

This does not mean that other optimization problems are secondary in some sense. Furthermore, the whole classes of optimization problems, such as the search of the optimal configuration of the three-dimensional elastic body [Banichuk, 1980] or the finding of the optimum surface contour of the thin shell [Banichuk et al., 2005] that remain outside the consideration are very important. But this kind problems, where the structure is described by partial differential equations are associated with mathematical ideas of another kind, and in this sense they are a little aside the subject of this essay. The only exceptions are the variants of the mentioned problems, which used the transition to discrete variables, for example, by applying the finite element method.

9.1. Inverse problem of structural mechanics

The classical theory of structures is traditionally focused on the analysis of stress-strain state. However, along with this trend there existed synthesizing elements directly related to such problems as optimal design and as solution of inverse problems of structural mechanics. These problems are not equivalent, since the solution of inverse problems of structural mechanics may have no aim to achieve the best (optimal) solution or to find a solution of some extreme quality for this system.

The stress-strain state of a system (for simplicity confine to bar structures), which search is a direct task of structural mechanics can be found if the following is known:

• topological scheme and geometric dimensions of the structure;
• supporting fixation and other conditions of constraint;
• types of cross-sections and their dimensions;
• physical model of the material behavior;
• external influences.
All these data are defined by some set of parameters and, if they are not all set, you can assume some elements of the stress-strain state, and the rest of its elements and unknown design parameters should be chosen in such a way that the problems be completely solved. Such a problem of structural mechanics is called the inverse one.

The inverse problem has often no clear answer and its terms can be satisfied by the entire set of values of wanted parameters. In these cases, the inverse problem conditions are often supplemented with the demand of choosing the best, in some sense, variant of solution, and there appears a problem of optimal design.

The most common solution is sought that is optimal from the viewpoint of material consumption or some other economic indicator. But in the inverse problem they do not necessarily proceed from the optimality condition; to find the unknown parameters they often use the terms of belonging to a certain class of the systems with some desirable properties (uniform strength, sustainability of distribution of elastic deformation in the system, the requirement of belonging of the frequency of natural oscillations to a certain desired range, etc.).

However, the unnecessariness is not a condition of the absence, thus there are inverse problems, where the unknown parameters are found for the system with extreme properties, and these are not necessarily the properties of minimum weight or cost. A typical example is the problem of search for the system with a set material consumption that can withstand the maximum load (the so-called warehousing problem). Or one can also give an example of the problem of searching for places to install additional supports (among positions allowed in advance) that maximize the stability loss load.

Fig. 9.1. Two viewpoints as to optimization problem
If we say about economic goals, there is a number of problems which complicate and clarify consistently the formulation, forming a chain: minimum amount (weight), minimum cost of the material, minimum cost of structures, minimum initial and maintenance costs throughout the life cycle. But it would be well to note that any kind of specification targets are not always significantly affect the results, so it is expedient to resort to very interesting observations made in the book [Hough, Arora, 1983] for the reaction of an engineer and mathematician to optimization results. As an example, the authors cite the well-known problem of optimal height of steel H-beam, for which it is shown that the change of optimal height $h_{opt}$ by 20% in either direction leads to the increase in weight per running meter by only 3-4%.

In terms of mathematics the discrepancy is reckoned along the axis $h$ and therefore points 0.8 $h_{opt}$ and 1.2 $h_{opt}$ are considered as a very rough approximation because of their considerable distance from the global minimum of the objective function (Fig. 9.1). And in terms of engineering the four-percent deviation from the global minimum of the objective function can be considered as a good approximation to the optimal project, and, when considering the result acceptance for the practice, the designer should not miss this opportunity, when choosing the solution of the design problem.

This idea was first apparently offered and widely considered by V.M.Gordeyev, who suggested that instead of finding the extremum point it is expedient to look for, and then to analyze in more detail the whole set of solutions that are adjacent to this point [Gordeev, 1974, 1987], [Shimanovsky, Gordeev, Greenberg, 1987]. Since most real optimization problems have a "slanting extremum", i.e. even noticeable deviation from the ideal solution does not much change the value of the objective function, it is possible, without leaving the space of decisions close to optimal, to consider additional conditions that are hardly formalized (e.g., the discreteness of certain parameters).
9.2. The beginning of the way

Attempts to solve optimization problems were made even in ancient times. Thus, even in times of Pythagoras it was known that a figure, which has the least ratio of perimeter to area, is a circle. In 1638 Galilei, who initiated the science of strength demonstrated parabolic shape of a beam of equal resistance [Galilei, 1638]. The idea of equal strength was used, which Galilei apparently thought quite natural for the considered problem, and which found numerous followers in the future.

In 1807 Thomas Young indicated that the optimal form of a hinged beam is useless because of its zero height on supports. He proposed to determine the shape of the beam at the site adjacent to the support with the help of a tangent of slope angle to the theoretical contour line of a bent beams, providing to some extent, Zhuravsky’s formula for the transverse force [Young, 1807].

But the first precisely worded problem of optimal design of bar structures was set and solved by Lagrange in 1770-1773. This was a problem on the column with the least weight, severely fixed at one end and loaded with the compressive force at the other end [Lagrange, 1773]

It was necessary to determine the shape of the column corresponding to the minimum weight at a given longitudinal force. Lagrange’s solution was wrong, and the error that it contained was eliminated only eighty years later in the work by Russian academician T. Clausen [Clausen, 1851], who has found optimal shape of the column (Fig. 9.2, a) in which the bar thickness tends to zero, when approaching the top, and the stress increases infinitely. To eliminate this particularity E.L. Nikolai [Nicolai, 1907] introduced
additional restrictions for permissible stress values. The obtained in this case thickness distribution is shown in Fig. 9.2, b.

M.G. Chentsov [Chentsov, 1936] and others continued the study of this problem. There was a detailed study of this problem for different types of rods and fixing conditions. In so doing both the mentioned task to minimize the weight of the rod at a fixed value of stability loss strength and the dual task to maximize the critical force at a given volume have been considered.

The Lagrange approach, when he formulates explicitly the problem of searching for the mechanical system, having some preset properties (in this case, minimal weight) allows us to consider Lagrange as a founder of a trend in structural mechanics focused on solving problems of synthesis. In addition, the use of Lagrange variational calculus to solve such problems permanently defined this method domination in solving problems of synthesis.

Works of Galilei and Lagrange were academic in nature and were not focused on technical applications. This situation has been retained for a long time, and synthesis problems were rather some exercises in the calculus of variations, which has been rapidly developing.

But there was one exception, which is associated with interest to calculation of bridge arches that stimulated research of arched

---

37 After the World war II, the Lagrange problem became popular in the United States. Clifford Trusdell, not knowing T.Clausen and his Russian followers offered the Lagrange problem the American scientists J. Keller and H. Weinberger. Both scientists coped successfully with the task. However, Weinberger’s work remained unpublished, while J. Keller has not only repeated Clausen’s solution, but also showed that the column has the shape of an equilateral triangle for optimal convex cross-sections.
systems. One of the most interesting questions of the theory of arches is the question of so-called rational form of the arch axis, i.e. the selection of the most advantageous geometric parameters of the arch axis and its cross-sections. This question has attracted the attention of engineers and generated great literature.

One of striking examples is the design of the bridge across the river, submitted in 1772 by I. Kulibin. He took the arch shape as a rope polygon, which was experimentally studied, believing that the arch of such form will work to the benefit of axial compression. It should be noted that this Kulibin’s judgment cannot be considered a pioneering one. More than a hundred years before Kulibin Robert Hooke, the specialist in experiments of the Royal Society, published the anagram which was deciphered as follows: "As a flexible line sags, the same, but in a reverse form, is a rigid arch" [Hooke, 1675].

Salimbeni made the first attempt to determine analytically the shape of the arch by the pressure curve in 1787, but his solution contained an error, since he believed that the pressure curve coincides with the arch axis at its any form [Salimbeni, 1787]. It took approximately sixty years until Villarceau solved correctly the problem of creating the arch, which axis coincided with the pressure curve [Villarceau, 1846].

At the beginning of the 20th century a lot of problems on optimal arches was solved. The question of defining the rational arch axis of the given span and rise was especially thoroughly studied, when the arch cross sections under the strength conditions are the least [Belzetskyy, 1904, 1907]. Further studies of this problem refined its formulation. For example, V.I. Rudnev advanced a new standpoint as to rational axis by establishing for arches of finite thickness the difference between the shape by the pressure curve along which $M = 0$, but $Q \neq 0$, the shape by vectorial curve along which $Q = 0$ but $M \neq 0$ and the shape by the intermediate curve. The flat arch shape can be searched by the funicular curve, along which $M = 0$ and $Q = 0$. For a number of integrated cases V.I. Rudnev derived the equations of
these curves belonging to symmetrical arches with a vertical load [Rudnev, 1930].

Studies aimed at finding a rational outline of the arch axis were continued in [Smirnov, 1950], [Filin, 1953], [Kiselev, 1953], [Gurevich, 1954] and others. But the problem on the arch was not the only among optimization problems that interested researchers in the late 19th and early 20th century. For example, engineer G.S. Semikolenov set a problem in respect of continuous beams "... to find a system of bridging arrangement to avoid the available disadvantages and to save, as far as possible, their significant benefits, savings on material" [Semykolenov, 1871]. In his "balanced" multispans beams he searched for places of optimal arrangement of hinges, thus improving cantilever-beam system of bridges proposed by Gerber [Gerber, 1866].

In connection with the development of rolled metal products assortments the problem of optimizing the cross-section shape of the bar subjected to bending caused great interest. The work by E.R.Patskevych [Patskevych, 1894] was one of the first published ones, where he justified for the first time a general method for analyzing profiles and created the basic theory of assortment. Using a specific moment of resistance, E.R.Patskevych has found that "the beam bending work is better, the larger is k, and not only strength, but also section rationality may be checked after k." Considering the profiles with equal specific moments of resistance, the author concludes that these profiles represent a particular case of profiles with equal specific resistivity. The equality of specific moments of resistance is determined by the equality of ratios of linear dimensions (specific points of resistance depend only on the shape of the profile). Later on the approach of E.R.Patskevych with some improvements was applied by F.S.Jasinski, who used it in the creation of Russian normal metric assortment [Jasinski, 1900]. In 1924 N.P. Puzyrevsky introduced an important concept of the theoretical weight of the structure, which played an important role in further studies [Puzyrevsky, 1924].
Quite close to this problem is a search for regularities that can be found, when studying the already constructed buildings of various types. So a systematic comparison of different bridge circuits in order to choose the best solution began with the work of E. Callington [Collington, 1865], the analysis was extended in the works by Heinzerling [Heinzerling, 1867], Dirksen [Dirksen, 1905], N.B. Boguslavsky [Boguslavsky 1907] E.O. Paton [Paton, 1914], N.S. Streletsy [Streletsy, 1925]. In particular, some general theoretical principles were advanced in the above work by M.S. Streletsy, based on the analysis of design decisions in respect of 320 bridges that were later developed in respect of other types of buildings both by the author himself and by scientists of his school.

This line of research is to some extent not theoretical but experimental, because we can assume that each real building is unique though not specifically planned experiment aimed at studying the behavior of the whole set of similar designs. The large amount of such experimental data allows using them to detect hidden patterns.

The process of design is largely unformalized experiment in terms of dealing with a particular problem. Since ancient times the data of these experiments have composed a fund for choosing rational actions, though such experiments, generally speaking, lead to random results and the time devoted to useless or ineffectual experiments, amount sometimes hundreds of years. For example, the search for a rational scheme for the bridge girder gratings was a long process that began with the Roman wooden arches, swept through the great number of strange shapes, such as the Bollman, Fink and Long [Perelmuter, 2016] schemes, and finally came to modern structures.

9.3. The origin of the theory

The above-mentioned studies were separate attempts to approach the creation of a more or less general optimization theory, the need for which was already felt in the 20’s of the twentieth century. Rudimentary forms of the theory were observed implicitly, for
example, in the works on the study of patterns inherent in previously built structures.

9.3.1. Equal strength and method of defined stress

This theory is based on the concept of equal strength or the structure, in which strength requirement for all reference sections is implemented in the form of equality (given stresses are implemented). That is the attainment of equal strength is announced as the aim of the structure synthesis.

Maurice Levy may be considered a predecessor of this approach. In the paper [Levy, 1874], he has found that the volume of bars of statically indeterminate equal-strength truss will be the same as that of the bar of statically determinable truss formed by elimination of redundant constraints. Besides, it was shown in his work that the theoretical weight of the equal-strength truss is proportional to strain energy.

It should be noted that Levi himself, when studying the properties of the equal-strength structure, never imagined that a search for such structures can be quite general approach to optimization, a something new method of structural mechanics.

Recall that the concept of equal strength was introduced by Galileo, who determined the shape of the equal-strength beams. He considered the case of bending of the cantilever beams under concentrated force applied to the free end, and it was shown that the equal strength condition is performed, if the beam height $h$ varies according to a parabolic law. As it was found, the problem on the shape of the beam of minimum weight provided that the normal stresses do not exceed the specified value $\sigma_0$ is reduced to the problem solved by Galileo. Thus, the equal-strength cantilever beam is simultaneously the beam with minimal weight. Other examples, when the condition of equal strength ensures the minimum weight of the structure were also found.
This circumstance has largely determined the interest in finding the structures of equal strength – the problem that makes sense in the case of one load.

The first general works in the field of search of the bar structures of equal strength belonged to A. Pippard [Pippard, 1922] and G. Heimann [Heimann, 1928]. They contain prescription describing a method of statically undeterminable trusses, which consists in giving forces in redundant (conditionally required) bars; allowing for these forces the sections of all other bars are chosen provided that stresses in them are equal to the limit value. Thus the truss (except for redundant bars) is that of the "total stress". Neither Pippard nor Gaiman noticed the possibility of obtaining the controversial solution, when a truss designed in such a way obtains the signs of forces in the redundant bars opposite to the set ones.

Full justification of the method and analysis of its both positive and negative sides were made in the classic work of I. Rabinovich published in 1933. The monograph [Rabinovich, 1933] had a huge impact on the further development of synthesizing direction in structural mechanics. He examined the issues of changing strains, forces and cross-section areas of bars; limits of such changes were established. Optimality criterion was the equal strength of the main bars. An important proposal on a possibility of creating a truss of the lowest weight, when using the prestressing (much later the result was rediscovered by Hofmeister and Felton [Hofmeister & Felton, 1979]).

The theorem of Maurice Levy was generalized; it concerned formation of statically indeterminate truss of the least weight through its transformation into a statically defined one by reducing forces in some conventionally required bars to zero.

The cases of contradictions under ineffectual giving forces in the redundant bars were thoroughly analyzed. The following problem that arose in connection with the method of defined stress proved to be nontrivial. Since it was offered to define forces in the redundant bars for statically undeterminable systems, one should know the
admissible limits of their change (going beyond these limits led to negative values of areas).

Some recommendations in this regard were given in 1938 by Huberyan [Huberyan 1938], who developed the method of specified stresses. Later on other researchers [Slyusarchuk, 1952], [Izraelit, 1956] also referred to the problems of admissible boundaries.

Another problem of the method of predetermined stress is connected with a necessity to operate on the absolute force values, since the values of the desired geometric characteristics of sections (areas, moments of resistance, etc.) depended on them. The "modular functions" of Y.A.Radtsig [Radtsig, 1946] owe their emergence to this problem; the problem of the most advantageous elimination of redundant constraints was solved with their help as well as with the help of the "sign inversion function" in the work of A.I.Vinogradov [Vinogradov, 1948] that determine locations of zero points of the moment curve in the optimal bar systems with predominant bend.

A.I. Vinogradov was the first to determine the question of analysis of defined stress for the effect of temporary loads and first introduced the concept of the most profitable influence lines. He continued to develop intensively the theory of inverse problems using his concept of a set of structures with a given outline of axes, the most profitable (in terms of minimizing the weight) distribution of internal forces being sought within this set. It has been proven that the minimum is achieved on a certain subset of the basic set, not only the system without redundant constraints, but also some statically undeterminable solutions without prestressing [Vinogradov, 1954] can be realized under these conditions.
In addition to Vinogradov’s research a number of works that develop the method of defined stress was performed by K.M. Huberyan. In his work [Huberyan, 1938] he proposed to hold the most profitable stress distribution, when developing a practical design model abandoning the most profitable force distribution. In his other work [Huberyan 1949] the proposed method of truss analysis by defined stresses was extended to multiply statically undeterminable trusses at the constant and temporary loading. Later K.M. Huberyan studied the application of the method of defined stress to the truss with cross lattice and found a significant simplification of the problem for the considered special case [Huberyan, 1951].

So, statically undeterminable truss of minimum weight can be realized only in exceptional cases, in contrast to structures which elements are in inhomogeneous stress state. Such structures can be always made those of equal strength, when changing the cross-section shape of elements [Goldstein, Solomesch, 1980]. True, we can obtain the structures of very exotic appearance.

An example of a portal frame design, its cross-section dimensions and orientation of its main axes being sought, is shown in Fig. 9.3 (a – frame scheme, b – a cross-section type c – the problem solution).

If we abandon the search of the section rotation angle, considering the direction of the principal axes as fixed, the equal strength solution can also be found, which turns out to be by 23.4% heavier.
Fig. 9.3. Optimal material distribution along the frame axis

The conditions of equal stability, sustainability of the specific potential energy of elastic deformation, etc., were used side by side with the equal strength condition as rationality design criterion.

9.3.2. Energy approach

Energy characteristics of optimal designs can be used as criteria ensuring minimum weight, and serve as a basis for the construction of the methods of their synthesis. The problem of search for the minimum is replaced by the problem of synthesis of the systems with predetermined properties.

For example, for totally stressed truss structure, in which the cross-section area of any bar is expressed through force in it $N_i$ and admissible stress $\sigma_0$ as $A_i = |N_i|/\sigma_0$, the strain energy is

$$U = \frac{1}{2} \sum_i \int \frac{N_i^2}{EA_i} \, dx = \frac{1}{2} \sum_i \frac{|N_i|}{E} \frac{1}{\sigma_0},$$

484
and its volume is

\[ V = \sum_i A_i l_i = \sum_i \frac{|N_i| l_i}{\sigma_0}. \]

Comparing these expressions, it is easy to see that \( U = kV \), where \( k = \frac{\sigma_0^2}{2E} \). Michell [Michell, 1904] was probably the first to notice the fact of proportionality. This relationship had become the starting point for A.I. Kefeli to calculate the least theoretical volume of a truss that was determined from the conditions of strain energy minimum [Kefeli, 1927].

Besides the condition of proportionality, it was found that the minimum volume can be obtained, when achieving the distribution of specific potential strain energy uniform by structure. Indeed, the factor constant for the entire structure \( k = \frac{\sigma_0^2}{2E} = \left(\frac{\sigma_0}{E}\right)\left(\frac{\sigma_0}{2}\right) = \frac{\sigma_0^2}{2E} \), constant for the entire structure, is equal to density of strain energy.

But studies of Kefeli had not been continued, only a decade later the book by Z. Wasiutynski [Wasiutynski, 1939] marked the beginning of a series of works in the theory of optimal systems, which were based on the relationship between material redistribution in the elastic linearly deformed system and the strain energy. Z. Wasiutynski set himself an aim to solve the problem of structure design as the problem on minimum of elastic deformation potential holding constant volume of material [Wasiutynski, 1950].

This trend also allowed studying the question of synthesis of engineering structures, a number of interesting work of a group of Polish scientists headed by Z.Wasiutynski (see, e.g., [Brandt et al., 1957], [Brandt & Ignaszak, 1958], [Biernawski & Grochowski, 1960], [Grycz, 1960] et al.) being dedicated to this question. The work by Masur [Masur, 1970] was of great importance for generalization of results found first for the truss structures; it proved
that the strength of elastic structure of a given weight is minimal, if the specific strain energy in the "design fibers" is constant throughout the system. Thus, the design fibers are considered as infinitely small cross-sectional areas, which stress state is affected by small changes in the structure parameters. Some works in optimization differed only in form from the studies, where optimization is connected with the strain energy..

So A.A. Komarov conducted his research proceeding from the idea that any structure is intended for the perception of some external loads and their transfer to the supporting fixing, and therefore the profitability of the designed force diagram will depend on the magnitude of transferred forces and on the length of the ways of this transfer [Komarov, 1952, 1965].

In this context a comparison of force diagram variants was made through the special design characteristics – its "force weight" $M$. It simultaneously takes into account the both force transfer qualities – the value and extent of internal forces in the structure. And the lower is its value, the more perfect is the force diagram.

However, as they say, new is the well-forgotten old. Almost a hundred years earlier K. Culmann [Culmann, 1866] proposed to distribute the truss nodes in such a way that to minimize the value $M = \sum |N_i| l_i$. But the value of the force weight $M$ is proportional to potential energy. Indeed, the expression for the energy

$$ U = \frac{1}{2} \sum_i \frac{N_i^2 l_i}{EA_i} = \frac{1}{2} \sum_i \frac{N_i^2 l_i A_i}{EA_i^2} = \frac{1}{2} \sum_i \frac{\sigma_i^2 l_i A_i}{E} $$

all stresses being equal to the limiting value, $\sigma_0$ takes the form

$$ U = \frac{1}{2} \sum_i \frac{\sigma_0^2 A_i l_i}{E} = \frac{\sigma_0}{2E} \sum_i \sigma_0 A_i l_i = \frac{\sigma_0}{2E} \sum_i N_i l_i. $$

Thus, the work of A. Komarov should also be referred to energy optimization trend. In general, the constant density criterion of
energy distribution used in this direction implements a minimum of weight in cases, when the dependence of quality criterion (objective function) and constraint function on design variables is linear.

9.3.3. Optimization as a problem of mathematical programming

The first method of finding of the plastic frame systems of minimum weight was proposed by Jacques Heyman [Heyman, 1951]. It is based on the study of possible fracture mechanisms and comparison of their respective weights. Since he dealt with continuous beams and frames, which are only affected by concentrated forces, the construction of such mechanisms caused no difficulties. It was also thought that the weight of the rod length unit is proportional to the ultimate plastic moment. A little later, W. Prager considered the case, when the weight of the unit of length is proportional to a certain degree (less than one) to the ultimate moment. [Prager, 1956].

Important theorems on the upper and lower limits for minimal weight of frame structures were proved by Foulkes [Foulkes, 1953], who had introduced the concept of a mechanism corresponding to weight. This mechanism characteristic is that the factors in the work equation at every ultimate plastic moment are the same as in the expression for the weight. The Foulkes’ theorems state that, if a mechanism corresponding to weight does not provide the appearance of statically admissible and safe distribution of forces, the weight of this frame will be less than that of the optimal structure. And, on the contrary, if the statistically admissible and safe force distribution is implemented in this mechanism, the weight of this frame is higher or equal to minimum weight.

Based on these theorems Foulkes came to the use of linear programming in optimal design problems. His approach consisted in finding a structure of minimum weight with nonadmission of plastic fracture of the structure. Under these conditions, a significant class of the problems of structural optimization can be formulated as a
problem of linear programming [Foulkes, 1954, 1955], only one loading condition being considered in this case.

In 1958 Pearson, when developing the theory of ductile fracture, addressed the problem of designing the trusses and frames of minimum weight with availability of several variants of loading [Pearson, 1958]. This work had played an important role because it used an approach, which served as the forerunner of three key ideas that later formed the basis for the development of modern methods of optimization of structures, working beyond the elasticity limits. They lie in the simultaneous search for both optimal design and critical patterns of appearance of plastic hinges.

Great influence on the development of modern methods of structural optimization had a work by Klein [Klein, 1955], in which it was shown that formulation of problems of nonlinear programming is possible for a rather general class of problems of design optimization, and a fundamental value of constraints as inequalities in the correct formulation of the problem of design optimization of the structures was recognized.

But the most general approach to solving optimization problems was proposed by Schmit [Schmit, 1960]. Within this approach the idea of using finite element analysis of structures relevant to the methods of nonlinear mathematical programming is introduced to create a system of automatic design. Schmit stressed the importance of allowance for nonuniqueness of various loading conditions and imposition of constraints on various forms of achieving the ultimate bearing capacity with the help of the system of constraints-inequalities in the proposed general statement of the problem.

Another important aspect considered in this problem formulation was the availability of constraints on the minimum and maximum sizes of structural elements. Besides, Schmitt pointed out that in contrast to popular belief the design of statistically undeterminable
structure of minimum weight with only stress constraints imposed on it is not necessarily such a structure, in which each element is fully loaded, at least, in one of loading variants. Over the next decade from 1960 to 1970 the approach based on nonlinear programming was applied to a variety of problems of design optimization and to development of the variants of implementing the algorithms of nonlinear programming.

Researchers tested methods of subsequent unconditioned optimization [Pope & Schmit, 1971], an approach that uses the Lagrange extended function [Gruver & Shafroth, 1976], the method of Rosen gradient projection and various modifications of steepest descent method [Gellaty & Gallagher, 1966], [Moses & Onoda, 1966] and other ways of solving the problem of nonlinear programming. The algorithm that was chosen by the author, had a significant impact both on the problem formulation, and on that considered its solution, since, as is known, the methods of nonlinear programming, in most cases, do not guarantee achieving the absolute minimum of the objective function through nonconvexity of the area of admissible solutions. This situation has led to the use of the methods of integer [Toakley, 1968] and dynamic [Kalaba, 1962], [Palmer, 1968] programming and different iterative methods [Reinschmidt et al., 1966], even for identical problems.

By the early 70's it became clear that the available opportunities of structural optimization at the global level, based on a calculation by the method of finite element analysis in combination with the method of mathematical programming, require extremely time-consuming solving of the problems of structural design of rather limited size.

It was believed that the way out of the situation is to use an alternative approach; its idea was proposed in the work by Prager and Taylor [Prager & Taylor, 1968] and became known as the "optimality criteria approach."
This method provides that you must first derive the conditions to be met by optimal design. Then an algorithm (as a rule iterative) is developed, aimed at finding a design that meets certain criteria; at the same time a certain local minimum is achieved. In this sense, the methods based on optimality criteria fall into the category of indirect optimization methods.

In pioneering work [Prager & Taylor, 1968], for example, they consider the problem of a search for the structure of minimum weight with a set value of virtual work of load $P(x)$ at the troughs $y(x)$.

As the virtual work of load has a set value $C$, the principle of minimum potential energy becomes the principle of minimum strain energy, and the latter as shown in [Prager & Taylor, 1968], is proportional to the structure weight. It was further shown that this kind of approach can be used, when considering the value characterized by the minimal principle of structural mechanics (such as was used in the above-mentioned problem of external forces). Indirect optimization methods were used in the problems of design of minimal weight systems [Venkayya, 1971], [Klusalas, 1972], [Dobbs & Nelson, 1976], as well as in a number of other problems, where one could rely on variational principles of structural mechanics. These include, for example, the following problems, in which the main frequency of natural oscillations [Grinev and Filippov 1971], the factor of ultimate load in the case of the plastic fracture [Chyras et al., 1974], the value

This direction also includes research on the problem of rational arrangement of constraints in the problems of stability and natural vibrations. One of the first works in this direction was the work by Bubnov [Bubnov, 1912-1914]. The rules for choosing places of imposing constraints for maximum shift of the first natural frequency or critical force were substantiated in the works [Nudelman, 1949], [Dolberh, 1951], [Smirnov, 1958], [Liakhovich et al., 1978]. They have justified their minimum number. It is shown that the installation of \( s \) additional constraints in the bar system can increase the value of the first natural frequency or critical force maximum to the \((s+1)^{th}\) value. In so doing the constraints should be placed at nodal points of the form of natural vibrations (stability loss) which corresponds to the \((s+1)^{th}\) natural frequency (critical force) of the system without additional constraints. The methods were proposed, which allow increasing the maximum value of the first natural frequency (critical force) with a minimum total rigidity of additional constraints.

Another formulation of the problem was considered in the work [Liakhovich, Plahotyn 1986]. According to this formulation you must choose the place of imposing the elastic discrete constraints of the number of possible ones and determine their stiffness in such a way that the first natural frequency (critical load) would reach a predetermined value, structural constraints would be kept to, and the weight or volume of material of additional constraints would acquire minimum value.

The solved problems are connected in some cases with the need to arrange additional loads on the structure, while in other cases – to remove some part of the load. In these conditions the change in frequency of natural vibrations should be within the set limits. The problem of full loading of the structure is aimed at the arrangement of the maximum possible additional load that does not remove the reduced frequency of natural oscillations out of the set limit. Therewith, the value of each additional load is limited.
Sometimes optimization criterion is constructed on the basis of intuitive reasoning. For example, the most important requirement, which should be met by any structure, is reduced to a necessity of keeping to strength criterion in each element. The strength restriction is among other restrictions imposed on the structure design. In practice, the strength criteria are satisfied using the concept of fully stressed structures; this concept has become one of the first optimality criteria.

It is difficult to denote the author of the idea of fully stressed structure, which intuitively seemed to lead to the system of the least weight and was implemented in many computation programs, but scientific analysis of this issue began only in the second half of the 20th century. The relationship between the fully stressed structure and that of minimum weight were of interest, since no obvious relationships between them were clearly visible. It was stated in the paper by Schmit [Schmit, 1958] that the minimum weight design cannot be chosen among fully stressed structures. But two years later it was shown [Schmit, 1960], that a fully stressed design is not necessarily the design of minimum weight, and even at the small number of loading conditions the full stress method may result in inefficient design.

And perhaps the first among the works that had clearly analyzed the relationship between the fully stressed structure and that of minimum weight was the work by R. Razani [Razani, 1965], where it was shown that the iterative method of search of the fully stressed structure is not always reduced to the structure of minimum weight and a necessary condition for the equivalence of two design methods is found. Later A.I.Vinogradov analyzed the convergence strength recalculation oriented to a search of completely stressed system and showed on examples that such recalculation cannot result in the system of minimum weight [Vinogradov, 1971].

The used assumption that in most practical designs the force element distribution is imperceptible to dimensions of cross sections of these elements, and that is why the considered algorithm for some
structures leads not only to non-optimal design, but to design with inefficient force distribution in structural elements. To get rid of this shortcoming, it was suggested to change design variables at each iteration by only a few percent, then the optimal design may be reached only after 4-5 iterations [Gallagher, 1973].

But truly great difficulties of solving big optimization problems could be overcome only by replacing the general problem of structure optimization by a sequence of relatively small explicit approximating problems of structural optimization. The implementation of this transition was achieved at the expense of coordinated use of the concepts, which foresaw the following:

- reducing the number of independent variables by uniting them into groups;
- reducing the number of restrictions taken into account at every stage by temporal rejection of inactive and redundant restrictions;
- construction of high-quality explicit approximations for the left constraint functions.

Generally speaking, it resembles the common practice of design, when a number of steps is implemented with gradual detailing and refinement of the problem.

In the early 80’s there was an attempt to create a new powerful method to minimize the weight of building structures, based on a combination of concepts of approximation with a dual method [Schmit & Fleury, 1980].

The analysis of this approach shows that the methods, based on generalized optimality criteria, and methods of mathematical programming form a single method for solving optimization problems of structures for a large class of the problems of size optimization of the structures of minimum weight. As shown by Fleury [Fleury, 1982], approaches to the theory of mathematical programming and approaches based on the use of optimality criteria, not mutually exclusive, and the difficulties inherent in conventional methods based on optimality criteria, are overcome by the combined use of the concepts of approximation and double approach.
9. 4. Synthesis of the schemes

You can trace how a range of the problems of structure optimization extended in the course of time. All the more complex constraints were accounted, problem formulations became more diverse, various methods of solving these problems were used. However a geometric schemes and topology of the structure were considered for a long period of time as set ones.

Only in a few cases, as it was in the problem of optimization of the arch axis, when the funicular curve was taken as the solution, which was previously considered as optimal, it was possible to find some parameters of the system geometry. The problem of determining the optimal value of the arch rise may be considered typical. And perhaps the first work which raised this question was the following study [Legay, 1900]. In 1925, I.M.Rabinovich has found that the most advantageous ratio of the rise to the span of a solid flexible filament or compressed arch, outlined by the pressure curve of its own weight, is expressed by the formula \( f/l = \sqrt{3}/4 = 0.433 \).

J.G.Panovko in his work [Panovko, 1934] presented the solution of the same problem, but took a load as the arch weight and entire filling above the arch. A number of other important cases were considered in the works [Kiselev, 1953], [Filin, Filalayeva, 1973].

9. 4. 1. Continuous cross-system

However the structure topology in these studies was assumed as preset and invariable. J.K.Maxwell was the first to break this tradition [Maxwell, 1869], he set the problem of finding out optimal configuration for a given truss under set loads and supports. Maxwell has derived a theorem, which states that in such a truss under given load and without the allowance for stability loss the condition is met

\[ V^+ R^- - V^- R^+ = \text{const}, \]

where \( V^+, V^- \) – the volume of tension and compression bars, \( R^+, R^- \) – ultimate strength of the material in tension and compression.
A. Michell, continuing the work of Maxwell, in 1904, proposed to replace the planar two-dimensional structure by the system of curved bars oriented along the principal stress trajectories [Michell, 1904]. As a result he obtained the structures that are statically determinable and cannot help but provide bearing capacity under loading with alternative forces.

Michell has proved that among all the truss structures which transfer a given load to supports, that one has the minimum volume for which the condition is fulfilled

\[ V^+ R^+ + V^- R^- = \sum_{i=1}^{m} P_i u_i, \]

where \( u_i \) – displacements in the direction of a force \( P_i \), and deformation module of bars is a constant value.

The form of Michell’s structure appears similar to conventional plastic slip lines (Henck-Prandtl grid) in plane-stress or plane-deformed state [Strang & Kohn, 1983].

The Michell’s structure usually includes the indefinably large number of infinitely long elements, so, in some cases they can be directly used in designing engineering structures. However, Michell’s structures may be useful in the design, especially with the dominance of one loading conditions and mainly the presence of restrictions on stresses.

Michell’s approach to designing the structures of minimum weight was further developed in the works of Hemp [Hemp, 1958] and Chan [Chan, 1960], but later this trend of research was gradually closed down, since the obtained solutions poorly met the practical requests (Fig. 9.4, a), though some schemes of this kind were used in the problems of fiberglass engineering.

495
But Michell’s approach remained attractive because it was little associated with a priori assumptions about the system structure and did not depend on the structure material. It is within such a problem statement that A.R. Rzhanitsyn’s research was performed, where the author makes an attempt of synthesizing approach to bar system analysis [Rzhanitsyn, 1949]. Using a concept of a virial of forces Rzhanitsyn established some general patterns characteristic of optimal systems, in particular, it was proved that in the optimal truss the volumes of compressed and tensile bars are equal to each other (this result was obtained in 1890 by Maxwell [Maxwell, 1890]. Minimal weight schemes were obtained for some problems (Fig. 9.4, b), like Michell’s schemes, they almost were not similar to real structures.

Approximately seventy years after Michell his theory was extended to systems subject to bending (raft foundations). William Prager and George Rozvany have first formulated the general theory of topology optimization, which is called "theory of configuration optimization" [Prager & Rozvany, 1977]. They applied it mainly to the accurate, analytical optimization of the structures similar to such cross-beamed cells, but it is also important for numerical methods and structures of the continuum type.

For Michell’s truss the per-unit cost function is defined as \( \psi = k|N| \) where \( \psi \) – weight per unit of length, \( N \) – force, \( k \) – given constant, and for the girder raft the per-nit cost function is recorded as \( \psi = k|M| \), where \( M \) – bending moment [Prager, 1974], that allows approaching the solution of the problem on optimal arrangement of such structures. Just as in Michell’s trusses the solution consists of the infinite number of densely spaced elements (truss-like continuum in the case of Michell’s problem and raft-like continuum in the problem of Prager). The problem of optimum arrangement is presented in detail in the monograph of Rozvany [Rozvany, 1976]. Some examples of obtained arrangements are shown in Fig. 9.5.
9.4.2. Using of special models of the finite element method

The next statement of the problem of structural synthesis proved more realistic. If a set of truss nodes with loads affecting them are given, then combining all these nodes by bars, making the equilibrium equation for all nodes and minimizing the amount of material we obtain the linear programming problem for the synthesis of the truss of minimum volume. This approach to the problem was first indicate by Pearson [Pearson, 1958]. Somewhat more general approach was developed in the article by W.S.Dorn, R.E.Gomory and H.J. Greenberg [Dorn et al., 1964], which used the concept of a set of admissible nodes (including unloaded), connected in pairs by bars
and by D.A.Matsyulyavichyus [Matsyulyavichyus, 1965], who suggested several modifications of the problem with allowance for many loads. A similar spatial problem was earlier considered by H.S.Chan [Chan, 1964].

The allowance for own weight was proposed in the work by Matsyulyavichyus [Matsyulyavichyus, 1969]. He has found the existence of a critical dimension of the structure; this dimension achievement causes such a large growth of its own weight, that the structure of material of a given strength is not able to accept it. A phenomenon of "weight instability" itself was apparently first described ten years earlier in the work of Vinogradov [Vinogradov, 1959].

Detailed analysis of the approach in [Dorn et al., 1964] has shown the dependence of obtained solutions on the initial density of the mesh nodes.

In the paper [Kohn & Strang, 1986] this feature is explained by the fact that at any point in the admissible geometric field with allowance for discreteness of the used mathematical models one of two possible "extreme" conditions is implemented: the construction material is either present of absent. Therefore, in the formulation of the problem of structural optimization of load-bearing units it was suggested to use specific porous materials [Komarov, 1984], [Bendsoe & Kikuchi, 1988]. This approach (homogenization method), which uses a solid deformable body with material characteristics variable in volume, admits a possibility of appearance of "transition" zones between "extreme" variants of the state of elastic medium in the design model.

In studies in this direction the elastic medium, inscribed in the admissible geometric area is divided into finite elements. The elastic medium compliance is taken as the objective function, and material mass serves as the restriction. The restrictions on equivalent stress, generalized displacements and critical forces of loss stability are taken
into account. Variations of design variables lead to degeneration of elements with irrational force transfer through them, and, on the contrary, the elements are distinguished in the structure that ensure rational transfer of forces. The topology of theoretically optimal structure is determined as a result.
First approaches to solving problems of topological optimization with regard for restrictions on stress were laid in the work [Xie, Steven, 1993], which examines the so-called "evolutionary" method of structure optimization. When using this method the ultimate value of reduced stresses RR gradually increases, under which the stressed elements, which do not exceed RR, are eliminated from the scheme (see Fig. 9.7.).

Another approach was based on solving the problem of optimizing the anisotropic properties of two-dimensional structural elements of locally-orthotropic material [Banychuk et al., 1984]. The best orientation of orthotropy axes of elastic medium on condition of minimum of a functional of integral rigidity is sought. The resulting distribution of orthotropy axes orientation can be used to form the structure and power force diagram, since the found lines of axes direction can be considered as lumped elements (such as stiffening ribs).

In general the problem of optimal design of thin-walled structures in its full extent is extremely complex and in some cases has no complete mathematical formulation for the structures made of composite materials [Banychuk et al., 1988], [Nemyrovskyy, Starostin, 1975], [Cherevatskyy, 1966]. This complexity is due to the fact that the problem on structure optimization belong to nonlinear problems of mechanics. The great number of forms used in the structure engineering, broad range of advanced requirements to them, and a variety of operating conditions also complicate the problem.

The most common optimality criteria are the requirement of minimum weight or minimum value (when the structure material is heterogeneous), because the objective function is characterized by integral functional. Studies made in this area are presented in the monographs [Armand, 1977], [Banychuk, 1986], [Banychuk et al., 1988] and so on. The analysis of the above literature shows that in most cases they considered only plane structures or cylindrical shells. This is determined by difficulties in solving the corresponding optimization problems.

500
As for the results of optimization, they often exhibit a rather sophisticated form of optimal designs with the outlines of elastic body limits unusual for traditional constructive solutions. Fig. 9.8 shows the distribution of thickness of a square plate with the hinged (Fig. 9.8, a) and fixed (Fig. 9.8, b) edge [Banychuk et al., 1980].

As part of the continual formulation of the problem of optimal design the optimality condition jointly with state equation and equations for combined variables form a closed nonlinear boundary problem concerning the variables of state, design and combined functions [Armand, 1977]. General analytical methods for solving such problems are not available, that is why the development of the theory of optimal design and efficient methods for solving applied problems are usually associated with discretization of the optimization problems. However, to narrow the space of design parameters (and thus simplify the problem), discretization is rather approximate (usually the structure is divided into 10 members), that reduces the reliability of the obtained results.

![Fig. 9.8. Optimal plate](image)

Fig. 9.8. Optimal plate

Another way to simplify the nonlinear structural optimization problems is the narrowing of the subspace of control functions and design parameters. This allows (in a number of cases) obtaining solution of optimization problem in analytical form or reducing to the problem of minimizing a function of several variables, but reduces the efficiency of optimal design. Thus, in [Gololobov, Ylyn,
1970], [Malkov, Strogino, 1971], [Nemyrovskyy, 1978], [Sable, Statnykov 1981] the optimum parameters of plates and shells are determined by the methods of mathematical programming at the expense of crucial choice of material anisotropy.

9.5. Calculation focused on optimal design

Popular programs of structure analysis by finite element method were developed without allowance for the features of design optimization problem. These features are as follows: the need to obtain data on the structure sensitivity to changes of parameters of the design model as well as the need for effective analysis of the great number of different designs having rather similar configurations. For example, in most programs, based on the finite element method, the stiffness matrix of $K$ system is constructed for a complete description of a specific project. However, given the need to solve the structure optimization problem, there arises the problem of allocation of invariant elements of the matrix $K$ [Bhatia, 1971]. And the first experience of solving the problems of optimal design showed that the structural analysis for choosing the optimal design is a specific task, when you need to analyze a lot of structures that are somewhat similar [Sergeev, 1975].

Sensitivity analysis plays an important role in the design of structures and in their optimization [Hough, Arora, 1983]. The gradients of design characteristics of the structure behavior (such as displacements, stresses, natural frequencies and normal modes) are found within the framework of sensitivity analysis as partial derivatives of these characteristics according to the design variable (e.g., cross-sectional area, thickness, spatial position of nodes). The urgent need for information on sensitivity analysis as an integral part of any modern program of the finite element analysis is dictated by the following considerations. When analyzing the sensitivity, the obtained valuable quantitative information can serve as a guide in the process of man-machine interaction, in addition, information on
sensitivity is the basis for constructing explicit expressions for the behavior characteristics of structures through the design variables.

It should be added that sensitivity analysis is also important in terms of checking the stability of the obtained optimal solution. There are cases, when even a small change in the problem parameters leads to a sharp change in the result. For example, in the work [Ovchinnikov, Bochkarev, 2003] it is shown that the optimal designs of the plates are very sensitive to thickness deviations from optimal ones, and for plates, designed to work in a mode close to the limit, even incomplete application of the design load at some part of the surface may lead to fracture.

The test of solution sensitivity to the change of parameters can indicate this feature of optimization result. This concerns not only the design variables that changed in the process of resolving, but those parameters which belonged to the set ones.

The main directions in research on the basis of structure design oriented to the optimal design of the structure, fall apart into three major categories:

1) methods of calculating the gradients of parameters of the structure behavior by design variables, that is the structure sensitivity analysis [Fox, 1965] and [Fox & Kapoor, 1968];

2) methods of construction of approximate solutions using a subset of state variables, chosen by the data of detailed analysis [Fox & Miura, 1971], [Noor & Lowder, 1974];

3) improving of the methods for partitioning into finite elements toward greater subordination of optimization problem of structural design [Bhatia, 1971].

The last two categories are associated with features of the design object and can hardly be universal.

**9.6. Untraditional optimization problem**

The traditional approach to structural optimization involves finding the most rational structure and distribution of material in the
construction, which is under a specified load or several loads of a
given set, discrete or continuous. The broader approach considers the
problems, which along with determining the optimal structural form
find out optimal distribution of the external force influences
[Matsyulyavychyus, 1968], [Litvinov, 1975], [Kashkovskaya, Liakhovich, 1998]. Variation of distribution and values of force
influence on the structure, when it is possible, allows improving
additionally the quality criterion without needless material
consumption. The variation of external influences may be possible,
for example, in the design of supporting structures of warehouse and
factory premises, where heavy load and equipment are arranged, as
well as in the design of ship structures, industrial cranes, etc.

One more feature of the traditional approach is the fact that the
main parameters determining the purpose and method of the object
use are considered as ordered "from command" and are not usually
subject to revision. For example, it is considered, that a design of
some building should be made, where some process should run, and
the question of organizing this process in the open air without
building any structure is not even discussed. However, it is
sometimes useful to break this tradition and consider the wider
setting of the problem of optimal design, while simultaneously
considering the problem of optimizing parameters of technological
process and design of the building or structure, where such a process
is implemented. The work [Perelmutter, Yurchenko, 2013] may be
considered as one of not numerous examples of this approach, which
examined the problem of optimal design of wind-driven electric
plant. In this work not only the features of the tower design
(geometry, the thickness of the shell), but the main operating
characteristics of the wind turbine (its designed capacity, plant height
and windwheel diameter) were attributed to the number of design
parameters.

The range of solvable problems is extended at the expense of
consideration of more diverse conditions of the structure service. For
example, large enough attention has been paid to the problem of
optimal design of the members of load-bearing units, their operating conditions being connected with the impact of aggressive working environments [Ovchinnikov, Pochtman, 1995]. The dynamic problems on searching the best decisions as for vibration protection, seismic insulation, damping of vibrations, optimization of the parameters of vibration suppressors, etc., are distinguished by great diversity (see, e.g. [Myzhydon, 1996], [Gordeyev, Dolgaya, Uzdin, 1997]).

But not only new problems are solved. The already solved problems are subject to a more rigorous analysis; besides, the techniques and approaches that have been found earlier and have become in some way sacred are also revised. For example, the well-known concept, which indicates the economic expediency of continuous growth of the unit capacity of industrial objects (for buildings – the principle of material concentration in the basic design) was profoundly studied in the work [Perelmuter, 2011]. It turned out that when allowing for the limitations that are determined by conditions of security, the above concept has applicability limit that is crucial.
References

Arman, J.-L.P. (1977), *Priloženije teorii optimalnogo upravleniya sistemami s raspredelyonnymi parametrami k zadacham optimizatsii konstruktsiy* [Application of the theory of optimal control of the systems with distributed parameters to structure optimization problems], Mir, Moscow, USSR.

Banichuk, N.V. (1980), *Optimizatsiya form uprugikh tel* [Optimization of shapes of elastic bodies], Nauka, Moscow, USSR.


Banichuk, N.V. (1986), *Vvedenie v optizatsiyu konstruktsiy* [Introduction to structure optimization], Nauka, Moscow, USSR.

Banichuk, N.V., Kobelev, V.V. and Rikards, R.B. (1988), *Optimizatsiya elementov konstruktsiy iz kompozitsionnykh materialov* [Optimization of structure elements of composition materials], Mashinostroyenie, Moscow, USSR.


Belzetsky, S.I. (1904), *Uprugaya lineynaya arka ravnogo sopotivleniya dlya davleniy proizvodyimych na vneshnyuyu poverkhnost arki sypuchim massivom* [Elastic linear arch of equal resistance for pressures on the external surface of the arch by loose massif], St.Petersburg, Russia.


Bubnov, I.G. (1912-1914), *Stroitelnaya mekhanika korablya* [Structural mechanics of the ship], Part1, Part 2, St.Petersburg, Russia.


Goldshtein, Yu.B. and Solomeshch, M.A. (1980), *Variационні задачі статики оптимальних стержневих систем* [Variational problems of statics of optimal bar systems], Izdatelstvo Leningradskogo universiteta, Leningrad, USSR.


Gurevich, Ya.I. (1954), “On the problem of rational law of the change of sections statically undeterminable bar systems”, *Trudy Khabarovskogo Institute inzhenerov zheleznodorozhnogo transporta*, Iss. 7, Transzheldorizdat, Moscow, USSR.


Izraelit, A.B. (1956), “On guarantee of positive solutions in calculation of statically undeterminable beams and frames me the method of set stresses”, *Trudy Vsesoyuznogo zaocnogo lesoekhnichekogo instituta*, no. 2, Moscow, USSR.


Kiselev, V.A. (1953), *Ratsionalnyie formy arok i podvesnykh system* [Rational forms of arches and suspended systems] , Gosstroyizdat, Moscow, USSR.

Komarov, A.A. (1965), *Osnovy proektirovaniya silovykh konstruktsiy* [Principles of design of force structures], Kuybyshhevskoe knizhnoye izdatelstvo, Kuybyshhev, USSR.


508

Nudelman, Ya.L. (1949), Metody opredeleniya sobstvennykh chastot i kriticheskikh sil dlya sterzhnevykh system [Methods for determining eigen frequencies and critical forces for bar systems], Gostekhizdat, Moscow, USSR.


Paton, E.O. (1903), Ves zheleznykhmostov dlya zheleznykh i lineinykh dorog [Weight of iron bridges for railway and linear roads], Izdatelstvo Politekhnicheskogo instituta, Kyiv, Russia.

Patskevich, E.R. (1894), Udelenyi moment soprotivleniya izgibu i yego primenenyi k raschotu metallicheskkh balok [Specific moment of resistance to bending and its application to calculation of metal beams], Tipografiya Erlich, St.Petersburg, Russia.


Perelmuter, A.V. (2015), Ocherki po istorii metallicheskkh konstruktiv [Essays in the history of metal structures], Izdatelstvo SCAD SOFT, Izdatelskiy dom ACB, Moscow, Russia.

Puzyrevsky, N.P. (1924), Shlyuzivnie vorota i propusk sudov cherez nikh [Sluice gates and boats passing through them], Stroyizdat. Moscow, USSR.

Rabinovich, I.M. (1933), K teorii staticheski neopredelimykh ferm. Zakony raspredeleniya usiliy: metod zadannykh napryazheniy; nachalnyie usiliya v staticheski neopredelimykh fermakh [On the theory of statically indeterminable trusses. Force distribution laws; set stress method; initial forces in statically indeterminable trusses], Transzheldorizdat, Moscow, USSR.
Rzhanitsyn, A.R. (1949), “On the problem on theoretical weight of bar structures” Issledovaniya po teorii sooruzheniy, Iss. IV, pp.252-265, Stroyizdat, Moscow, USSR.


Rudnev, V.I. (1930), On rational shape of a continuous elastic arch in connection with modern methods of erection”, Trudy MIIT, Iss. 15, Transpechat, Moscow, USSR.


Slyusarchuk, F.I. (1952), “On regular growth of statically indeterminate trusses by the method of set stresses”, Trudy Novosibirskogo instituta inzhenerov zheleznodorozhnogo transporta, Iss. 8, Transzheldorizdat, Moscow, USSR.


Smirnov, A.F. (1958), Ustoichivost i kolebaniya sooruzheniy [Stability and vibrations of structures], Transzheldorizdat, Moscow, USSR.

Sobol, I.M. and Statnikov, R.B. (1981), Výbor optimalnykh parametrov v zadachakh so mnogimim kriteriyami [Choice of optimal parameters in the problems with many criteria], Nauka, Moscow, USSR.

Streletsy, N.S. (1968), Zakony izmeneniya vesa metallicheskih mostov. Izbrannye trudy [Law of weight change of metal bridges. Selected works], Stroyizdat, Moscow, USSR.

Streletsy, N.S. (1926), “Zakony izmeneniya vesa metallicheskih mostov”, Sbornic trudov byuro nauchno-tekhnicheskogo komiteta NKPS, Coll. 8, Transpechat, Moscow, USSR.

Filin, A.P. and Filalayeva, E.S. (1973), Ob otyskaniy optimalnoi osi tryokhsharnirnoi sistemy pri nabote yeyo na neskolkikh variantakh nagruzki [On finding of optimal axis of three-hinge system under its work with several variants of load], Izdatelstvo KGU, Kazan, USSR.

Filin, A.P. (1953), “Problems of rational design of bridge arches”, Trudy Khabarovskykh instituta inzhenerov Zheleznodorozhnogo transporta, Iss. 3, Transzheldorizdat, Moscow, USSR.
Hog, E. and Arora, J. (1983), *Prikladnoye optimalnoye proektirovaniye: Mekhanicheskie sistemy i konstruktii*, [Applied optimal design: Mechanical systems and structures], Mir, Moscow, USSR.


Shimanovsky, V.N., Gordyev, V.N. and Grinberg, M.L. (1987), *Optimalnoye proektirovaniye prostranstvennykh reshetchatkh pokrytiy* [Optimal design of spatial net covers], Budivelnik, Kyiv, USSR.


Culmann, K. (1866), Die graphische Statik, Zurich, Switzerland.

Dirksen, (1905), Hilfwerke für das Entwerfen der Brücken mit eisernen Unterbau.


Galileo Galilei. (1638), Discorsi e demostrazioni matematiche intorno a duonuove scienze attenenti alla mecanica e I movimenti locali, Appreviso gli Elseviri, Leida.


Gerber, G.H. (1866), Balkenträger mit freiliegenden Stützpunkten Bavarian patent.
Heimann, H. (1828), Beitrag zur Berechnung statisch unbestimmter Fachwerke, Berlin, Germany.
Heyman, J. (1951), Plastic design of beam and plate frames for minimum material consumptions, Quarterly of Applied Mathematics, Vol. 8, no. 4, pp. 373-381.
Hooke, R. (1678), Lectures de potentia restitutiva, or of spring explaining the power of springing bodies, John & Martin Printer to the Royal Society, London, UK.
Klusalaas, J. (1972), Minimum weight design of structures via optimality criteria, NASA Technical Notes, D-71715.
Legay. (1900), Memoire sur la trace et le calcul des voates en macronerie, Annales des ponts et chaussées, no. 4, pp.19-34.
Levy, M. (1874), La statique graphique et ses applications aux constructions, IV partie, Paris, France.
Maxwell, J.C. (1869), Scientific papers II, Univer. Press, Cambridge, USA.
Pearson, C.E. (1958), Structural design by high-speed computing machines, Conference on Electronic Computation of ASCE, Kansas-city, USA.


Salimbeni. (1787), Degli archi e delle volte, Verona, Italy.


Young, T. (1807), Course of Lectures on Natural Philosophy.
Essay 10

STATICS AND STABILITY
OF THIN-WALLED BARS
There could be no fairer destiny for any physical theory than that it should point the way to a more comprehensive theory in which it lives on as a limiting case.

A. Einstein

It (the ugly duckling) ... remembered how he had been laughed at and cruelly treated, and he now heard everyone say he was the most beautiful of all beautiful birds.

H.Ch. Andersen
In structural mechanics a body, which one dimension is much more than two others \((L \gg d)\) is called “a bar”. As to other two dimensions, they can have a value of the same order, and then we say about a massive bar or, these dimensions are of different order \((d \gg t)\), and we are dealing with a thin-walled bar \((L \approx \text{length of the bar axis}, d \approx \text{overall dimension of cross-section}, t \approx \text{wall thickness})\). A behaviour of bars with open and closed profile of the cross-section are markedly different, the former noticeably differ from the massive bars.

The second of the above inequalities is also characteristic of shells. Thus, a thin-walled bar for which both inequalities \((L \gg d, d \gg t)\) are implemented, partially inherits and unites the parent properties both of the bar and shell.

It was found long enough that the main difference in the behaviour of massive and thin-walled bars relates to their torsion work. The known solution of the Saint-Venant problem on torsion of a prismatic body describes accurately the behaviour of a solid bar of circular cross-section, relative to well-massive bars of other cross sections, and was at variance with observed behaviour of thin-walled bars.

**10.1. Thin-walled elastic bar**

**10.1.1. Classical period**

The work of Rudolf Bredt, who considered peculiarities of free torsion of bars with noncircular tubular section [Bredt, 1896] may be attributed to the first research of this problem. Based on geometrical considerations and using the hydrodynamic analogy in which a flow of running forces \(T(s) = \tau(s) \cdot t(s)\) is compared with a fluid flow constant in length (Fig. 10.1), he concluded
that the shear stress $\tau$ cannot vary considerably in the wall thickness and in design it can be understood as the average value of $\tau$, and thus shear stress is inversely proportional to the wall thickness.

Hence, the rule has been derived:

“For a closed thin-walled profile with small wall thickness $t$ the shear stress is considered to be uniformly distributed over the wall thickness and is defined by the formula

$$\tau = \frac{M_t}{2At},$$

where $A$ – the area of the figure, enclosed by the median line of the section.”

This rule was quite practical, though related to only one case of free torsion of single-circuit closed thin-walled bar.

![Fig.10.1. Bredt’s scheme](image-url)

Fundamentally important step in the theory of thin-walled bars was made by S.P. Timoshenko, when results of his research of H-beam torsion, one of which sections remains flat, was published [Timoshenko 1905-06].

That was the rare case, when we can indicate the exact time of the discovery. In summer 1905 S.P. Timoshenko worked on probation in the Ludwig Prandtl laboratory at Göttingen.
In 1899 Prandtl solved the problem of stability of the flat shape of strip bending, practically the first problem of stability of thin-walled bar.

Timoshenko wrote in his memoirs: "Prandtl proposed to extend his own thesis. He examined lateral buckling under the bending of narrow rectangular beams, but it was certainly more important to study the lateral stability of the H-beam for practical application. It is necessary to start with H-beam torsion in this case. It first became clear that Saint-Venant's principle is not applicable to solve the problem. The twist angle depends not only on the torque magnitude and on the beam torsional stiffness, but also on the way of fixing the beam ends. If the beam end is fixed, it is obvious that H-beam flanges are bent, and this bending should be allowed for. About two weeks passed before I realized how to allow for this bending. I realized that a torque is balanced by such stresses as in ordinary torsion, which are summed up with the moment formed by shearing forces arising under the bending of H-beam flanges" [Timoshenko, 2014].

There is the clearly indicated difference between free and restricted torsion that is demonstrated by Fig. 10.2, borrowed from the later S.P.Timoshenko’s work [Timoshenko, 1910]. The obtained theoretical solution of the problem has been verified experimentally, when twisting angles measured in experiments coincided well with the theoretical values calculated by formulas.

It was clearly revealed in the S.P.Timoshenko paper that the H-beam bar torsion appears as the bending of flanges, and so, it is accompanied by formation of a self-balanced normal stress system in cross-sections. This effect of appearance of normal stresses in the cross-sections of bars being subjected to torsion, is important for all thin-walled bars with open profile and to a lesser extent for those with closed profile. It was found that two additional factors (normal
and tangential torsional stresses) appear in the thin-walled profiles under limited torsion, which are not considered by the standard formulas of the strength of materials.

Fig. 10.2. Free and restricted H-beam torsion (figure of [Timoshenko, 1910])

The next important achievement in the theory of thin-wall bars was finding the flexural center, after Bach had published his experiment results in 1909 [Bach, 1909], [Bach, 1910]. Through experiments with a metal beam of channel-type cross-section, Bach has found that the lateral load acting parallel to the channel wall and passing through its gravity center, causes torsional strains jointly with bending strains. The extension deformations of four outer fibers of the channel at arbitrary load position are not governed by the law of plane sections. Torsional strains of the channel in Bach’s experiments were significantly lower, when the lateral load passed through the channel wall axis than in the case of load application at the center of gravity. Finding experimentally the deviations from the law of plane sections, Bach explained this deviation by the cross-section asymmetry. The explanation was wrong, but the experimentally established fact initiated further studies.

After the Bach and Timoshenko works the problem of torsion of thin-walled beams accompanied with bending of certain elements was not elucidated in press over years of time. But in 1921, i.e. 12 years after Bach's experiments, there appeared the work by Maillart [Maillart, 1921], devoted to the question of bending and torsion of thin-walled metal beams. In this work the author, when analyzing Bach’s experiments, notes that the deviations from the law of plane
sections under torsion accompanied by the bending of certain elements can also take place in symmetric profiles.

Maillart, in his subsequent articles, presents, besides experimental results, calculated data concerning determining the flexural center [Maillart, 1922]. These data were obtained by him on the basis of the method of S.P. Timoshenko. Maillart determines the flexural center as the point at which the sum moment of the resultant elementary tangential stress is balanced (Fig. 10.3).

![Diagram showing flexural center determination](image)

**Fig. 10.3. On determination of flexural center**

An experimental work of S.A. Bernstein [Bernstein, 1927] appeared in 1927. The author, confirming the results of Bach’s experiment, noted a significant deviation of normal stress distribution nature in the cross-sections of truss booms of open bridges from the law of flat sections and called this phenomenon "deplanation."

In the period from 1921 to 1926 the works of Zimmerman [Zimmermann, 1922], Eggenschwyler [Eggenschwyler, 1921], [Eggenschwyler, 1924] and Weber [Weber, 1926] were published. The latest work, in addition to the method for determining the bending center, generalizes Timoshenko’s results about H-beam torsion, indicating the method of determining additional torsional normal stresses for any profile with two flanges (H-beam with similar and different flanges, channel-type and Z-type). In the same paper Weber noticed the relationship between the flexural center and torsion center, i.e. the section point that does not move under torsion. He believed, by mistake, that the both points coincide under the torsion accompanied by bending of the profile flanges.
In 1929 Wagner outlined the features of the theory of thin-walled bars of arbitrary form of the open profile [Wagner, 1929]. When deriving his formulas of additional normal torsional stresses, Wagner used the law, similar to the law of sectorial areas derived by V.Z.Vlasov in 1936 for the profiles of arbitrary outline [Vlasov 1936]. When considering torsion deformation, Wagner believed that the torsional center under loss of stability coincides with the bending center. In fact, the torsion center does not usually coincide with the bending center. They coincide only in one particular bar cross-section case, when the bending center coincides with gravity center of the cross-section. Probably Ostenfeld [Ostenfeld, 1931] was the first to notice inaccuracy of Wagner’s results and obtained accurate solutions for T-section, as well as for the corner and channel-type cross-sections.

In 1936 Frederick Bleih and Hans Bleih published the theory of bending, torsion and stability of thin-walled open bar with polygonal profile [Bleich F. & Bleich H., 1936]. The authors of this article, using energy method in the stability problem, obtained a system of three differential equations related to the case of central compression.

However, they proceeded from the law of flat sections and replaced normal stresses, defined in cross section, by the resultant force, taking it as a concentrated force applied to the gravity center. As a result of this change one of three roots of the relevant determinant equation was lost, and inaccurate results were obtained for the other two roots.

The theory of bending, torsion and stability of a thin-walled bar of arbitrary profile proposed by Kappus in 1937 [Kappus, 1937] a year later than by V.Z.Vlasov.

A statement of the problem about thin-walled bars and its solution, expounded with maximum fullness in the book by V.Z.Vlasov, which was published in 1940 [Vlasov 1940].
V.Z.Vlasov departed from his own theory of cylindrical shells, but he simplified general equations for a thin-walled bar, introducing two kinematic hypotheses: a hypothesis of the absence of displacements of the middle surface and hypothesis of invariability of the cross-section shape. It was found that the deplanation under compressed torsion differs only in scale from deplanation corresponding to free torsion of the same bar (the scale varies from section to section).

A summand initiated by bimomentum and determined by the law of sectorial area is present in the formula for the normal stress besides three ordinary terms.

The developed theory allowed the researcher to make a comprehensive solution of the problems on the torsion work of a thin-walled bar, on flexural-torsional form of stability loss and on vibrations of thin-walled elastic bars, and to develop methods for calculation of bars with elastic and rigid constraints and methods for calculation of bars under transverse loads.

Somewhat unusual concepts of bitorque, sectorial inertia moment, etc., have first emerged in this theory. But, as life has shown, these concepts soon became habitual. The possibility of application of the Saint Venant principle was studied in detail and it was shown that it was limited here. In particular, V.Z.Vlasov discovered a twisting effect of the thin-walled bar under the effect of longitudinal loading. For example, for a Z-like profile under wall extension the elongation of its outer fibers leads to flange bending, creating the twisting of profile (Fig. 10.4).
In succeeding years there appeared a huge amount of literature, devoted to discussions on the principles of V.Z.Vlasov’s theory, development of methodological variants of its construction, e.g., the works of Dzhanelidze [Dzhanelidze, 1943, 1944]); the possibility of rejection of some hypotheses were analyzed in these works.

The hypothesis of the section outline invariability did not cause specific objections. The fact that the supporting elements such as stiffening ribs, diaphragms, etc., really create conditions similar to those approved in this hypothesis. Much more attention was paid to the hypothesis of shearing absence in the middle surface.

The most reasonable study of accuracy of this theory was made by A.L.Goldenweiser [Goldenweiser, 1949]. Based on the shell theory, he has shown that the hypothesis of outline undeformability is performed too accurately even for short bars, while the hypothesis of shearing absence does not lead to significant errors only if \( d/L \sim t/d \ll 1 \).

The problem of design analysis of compressed torsion of a thin wall-bar of closed profile slightly delayed in their development. Bredt solution satisfied practical engineers, while researchers were struggling with the riddles of bars of open profile. Only in 1926 E. Reissner considered the compressed torsion of a rectangular box-like bar [Reissner, 1926], and then for many years, this particular case was, in fact, the only object of research in this field.

The first in the world literature publication on compressed torsion of thin-walled bars with closed profile refers to 1932 and belongs to V.N. Belyaev [Belyaev, 1932]. He considered a rectangular box that consists of four relatively strong zones, four thin walls and a number of diaphragms. V.M. Belyaev suggested to simplify the solution, considering a wall as such that does not work on normal stresses (in the section perpendicular to the design axis), and is able to perceive only tangential stresses. At the same time it was decided that the corner zones work only on longitudinal forces (Fig. 10.5). V.M. Belyaev obtained convenient chain system of equations for
determining the redundant unknowns ("the equation of three axial forces").

Fig. 10.5. Model of thin-walled bar with closed section

The importance of V.M. Belyaev’s work consists in identifying and analyzing the phenomenon of compressed torsion in the structures with closed profile, but the proposed method is directly concerned only four-zone structures but does not make it impossible to investigate the compressed torsion in a more general case.

Aviation industry gave impetus to new research. It was necessary there to predict in detail the behaviour of monocoque airplane wing under torsion.

Creation of applied theory of thin-wall bars with closed profile of arbitrary shape owes mainly to the works of A.A. Umansky. He proposed the original method for design of thin-walled bars with rigid closed profile and, basing on the bimoment theory, he considered a number of new problems concerning the calculation of a paired flat bar structures, which he called bistructures.

This study, performed by A.A. Umansky [Umansky 1939] initiated the study of large problem of compressed torsion for the general case of arbitrary closed profile structure. A.A. Umansky advanced a proposal to describe deplanation under compressed torsion using the same law as in the case of free torsion (but the measure of deplanation displacements was variable along the
structure length) and to consider the cross-sectional shape in its plane as such subject to distortions (Fig. 10.6).

![Fig. 10.6. Deplanation and bimoments of a bar with open and closed section](image)

A rather general solution of the problem of compressed torsion of a bar with arbitrary closed profile was obtained on this basis.

In further work of A.A. Umansky [Umansky, 1940] the said ratio was replaced by another, and the second edition of the A.A. Umansky theory was more accurate. Quite similar publications of these works appeared much later but without reference to A.A. Umansky. The first version of the A.A. Umansky theory was presented in the works of Kármán and Christensen [Kármán & Christensen, 1944] Raithel [Raithel, 1963] and Rudiger [Rudiger, 1964]. The second variant of the A.A. Umansky theory was expounded in the publication of Grasse [Grasse, 1965].

The book [Dzhanelidze, Panvko, 1948], in addition to the main results relating to the justification of bimoment theory of bending torsion of the bars of open profile, contains the methods for determining stress and strain of the compressed torsion of a bar with closed profile.

They also examined the ways of solving the problems on compressed torsion based on approximate replacement of a thin-walled bar by a system of thin zones that work on longitudinal forces and bound by casing that works on a shift only (V. Belyaev scheme,
Fig. 10.5). In 1950 Benscoter [Benscoter, 1950] came to this scheme, then it was developed effectively by Argyris and Kelsey [Argyris & Kelsey, 1960].


The study of the work of a thin-walled bar beyond the elasticity limit was started the same year. These works include the article [Rzhanitsyn, 1941].

O.I. Strelbitskaya [Strelbitskaya, 1954], [Strelbitskaya 1956] conducted studies at the S.P. Timoshenko Institute of Mechanics of the National Academy of Science of Ukraine. Her studies, confirmed by experiments, were of great practical importance.

10.1.2. Back to sources

The desire of a more rigorous substantiation of the methods of practical analysis which is typical of the last time, has created a series of works devoted to careful analysis of hypotheses, assumed as a basis of the applied variants of the theory of thin-walled bars.

Typical in this respect we can call the work [Rodriguez & Viano, 1997] in which V.Z. Vlasov’s theory was obtained as the asymptotic approximation of three-dimensional model of the prismatic body, in which the thickness of cross-section elements tends to zero. Works of the mentioned trend are often initiated not only by the desire to "bring order in basics" but also by the problems of constructing rather universal algorithms for programming the problem with the
aim of its numerical solution. We cannot often rely on engineering intuition and informal analysis, which help choose one or another hypothesis, and it is necessary to have clearly formalized rules of behaviour.

The discrepancy between the theory of thin-walled bars of open profile, which does not allow for shear deformations, and the theory of thin-walled bars of closed profile, developed on the basis shears, did not suit researchers for a many years. Various attempts were made to reconcile these approaches which allowed, in one way or another, for the effect of shear deformation on the work of a thin-walled bar. A notable result was recently obtained by V.I.Slivker [Slivker, 2005], who proposed to present shearing stresses as a sum of two components: bending shear stresses generated by transverse forces, and torsion shear stresses, caused by the moment of compressed torsion.

It was further suggested to neglect the shear bending stresses, referring it to the minor category, but at the same time remain shear twisting stresses. This semi-shear theory is universal to calculate both thin-walled bars of open profile (based on the V.Z.Vlasov theory) and the bars of closed profile (based on the A.A.Umansky theory), allowing for the similarity of the respective differential equations of torsion and energy functionals.

The hypothesis on invariability of the cross-sectional shape is an important assumption of the theory of thin-wall bars. However, in some cases this hypothesis is in contradiction with observational data. In the early 20th century that fact drew attention at the study of behaviour of the curved pipes, when calculation results of ordinary theory of bars diverged strongly from actually observable deformation pattern. The paper by T. Karman [Karman, 1911], which is considered now as a classic one, was the first work, where the author revealed a cause of a noticeable difference between the theoretical and experimental data.
In the course of time, researchers began studying the results of rejection of the hypothesis on cross section nondeformability, when the question of local stability of steel structures made of thin-wall curved profiles, acquired urgency.

Mathematical difficulties of solving this nonlinear problem deferred research in this field, and only the use of numerical analysis techniques, including finite element method, made this formulation of the problem a practical one. For example, in the work [Nedelcu, 2009] the author considers the equilibrium of a thin-walled bar with nondeformable and deformable cross section and touches upon the questions of stability.

Numerical studies of the behaviour of thin-walled bars are most often performed using the finite element method. It is also used to solve nonlinear problems which analytical solution is practically impossible. The paper by R. Barsoum and R. Gallagher [Barsoum & Gallagher, 1970] may be considered as one of the first works of this type, which dealt with solution of the stability problem for the flat form of equilibrium of a thin-walled bar of open profile under pure bending. The behaviour of a thin-walled bar under large displacements and other questions of nonlinear behaviour were investigated in other studies [Chen & Blandford, 1991], [Meek & Lin, 1990], [Conci & Gattass, 1990].

10. 2. Stability of equilibrium

10. 2. 1. Action of longitudinal force

As was noted above, the theory of thin-walled bars originates from the problem on stability of a plane form of H-beam bending. But serious study of the stability of such bars under the action of longitudinal force began, in fact, in the works by V.Z. Vlasov. Within his approach the problem on stability of the bending plane form becomes a special case of general theory of equilibrium stability of thin-walled bars.
Some fundamental differences were found in behaviour of the thin-walled bar from the Bernoulli-Euler usual bar.

Firstly, it appeared that the stability loss phenomenon may appear at noncentral action of longitudinal force both in the case of compression and tension. A bar may lose stability at any position of force in the cross section under compression; in case of tension the stability loss phenomenon can occur only if the longitudinal force is applied beyond the area of stability, which is a circle regardless of the bar cross-section form.

Secondly, bending and torsional forms of stability loss of equilibrium in a general case of centrally compressed bar are not separated. And if the coordinates of the bending center are not equal to zero (bending center does not coincide with the section gravity center), the Euler bending form of stability loss under the central compression becomes impossible. The natural form of stability loss for such a bar is flexural-torsional at which the critical force will be of less value than the force found when following the usual theory of longitudinal bending.

The equilibrium stability equation of thin-walled bars, obtained by V.Z.Vlasov, caused a lively discussion at that time. The main motive of this discussion was a way of deriving these equations. When constructing them V.Z. Vlasov did not proceed from the conditions of equilibrium stability, leading to homogeneous equations, but from the deformation equations, rejecting their right parts in determining critical forces. The mathematically perfect way of constructing the stability equations for thin-wall bars, leaving no loopholes for further debate about the severity of deriving these equations and based on variation stability criterion, was given only in 1965 by V.Bolotin [Bolotin, 1965].

10. 2. 2. Local stability

With a decrease of thickness of thin-walled bar elements the features of behaviour begin to manifest which cannot be described by V.Z.Vlasov’s classical theory based on the hypothesis on
invariability of the bar cross-section outline and only three forms of stability loss (bending, torsion, flexural-torsional) are considered.

The possibility of distortion of the cross-section outline of a thin-walled bar under compression and bending is manifested as a loss of local stability when separate plates that make up the outline are buckling. Therein the contact lines of contiguous plates continue to be straight (Fig. 10.7, b).

Later on J.B.Dwight [Dwight, 1963] in his experimental work and M.L.Sharp [Sharp, 1966] in his theoretical studies noticed a special local form of stability loss associated with a simultaneous loss of stability of several elastically connected plates, jointly forming a bar (Fig. 10.7, c). It was called a distortion.

Works on optimizing the form of the thin-walled bar cross-section showed that the distortion is often the dominant form of stability loss for such cross-section. This fact has stimulated the research of distortions and development of appropriate methods of analysis. Hancock and Lau [Hancock, 1985], [Hancock & Lau, 1987] developed and popularized the method of finite bands as a research tool. They studied the role of boundary conditions and performed a comparison with the data of experiments, which identified the distortion.

Fig. 10.7. Stability loss of thin-walled bar: a – flexural-torsional, b – local, c – distortional

Experiments have shown that the bearing capacity of a thin-walled bar at the moment of local buckling cannot be exhaustive. In order to evaluate the bearing capacity, it is necessary to investigate
the supercritical behaviour of a compressed bar. In so doing the following problems are solved:

1) determination of the stress distribution law in the supercritical phase by the element width to define "efficient", part of plate which takes up loading;

2) study of the general stability and analysis of breaking load of a bar with a weakened ("reduced") section.

The former problem – determination of the effective width (or "reduction factor") of a separate plate after local buckling was first reviewed by I.G.Bubnov [Bubnov, 1908]. Later T.Karman [Karman et al., 1932] presented an engineering method of allowance for the impact of local stability loss based on the concept of "effective cross section", when the reduction of bearing capacity of a single plate, as a result of its buckling, was replaced by conventional "exclusion" of its part from the work. This technique, with regard for modifications proposed by Winter and based on the experiment results [Winter, 1947], is still used today.

Various researchers have obtained different formulas for determining the reduction coefficient. It should be noted that the exact solutions were made for some plates with idealized boundary conditions (hinge support or stiff fixing), and for thin-walled structures such solutions can only be considered as approximate ones. In this regard, the work of Koiter [Koiter, 1976] is of interest, where a simple formula is obtained for a lower bound of the reduction factor.

The problem on bearing capacity of thin-walled bars composed of plates after local buckling were probably first considered by Bijlaard and Fischer [Bijlaard & Fischer 1954]. Compressed pillars of the flange and square sections were studied. It has been shown that additional (in respect of the local form) displacements of a thin-wall bar element determined by the overall bending are similar to the other local form.

First works dealt with only unilateral interpretation of interaction of buckling forms, that is with the allowance only for the effect of
local buckling on further general loss of stability. This approach may be sufficient in the case, when the general buckling stress is much higher than critical stresses of the local form, and the both stages of the buckling process can be divided.

In the general case, the local and general buckling forms, can influence one another in a complicated way, and this "mutual" influence is an important factor determining the bearing capacity. The neglect of this behaviour feature of the compressed thin-walled bars could lead to very dangerous consequences in practice. For example, it has been suggested that the interaction of general and local forms of stability loss became a possible cause of a number of great accidents of steel bridges in Vienna (1969), Milford (1970), Melbourne (1970), Koblenz (1971). The failure occurred at loads of 30% and more below the theoretical ones (wherein the design critical stresses of the general and local buckling of ribs were close).

The work by Neut [Neut, 1969] proved an important step in the development of the theory of related buckling of the plate-bar systems; he considered the box-type pillar as a model consisting of two bearing flanges-plates (the rigidity of side plates is neglected). Owing to allowance for local imperfections, the limit load can be both below and above the critical value of the local load form. The novelty of the work [Neut, 1969] consisted, first of all, in finding the equilibrium instability of the pillar at the bifurcation point in the case of proximity of critical stresses of Euler’s and local buckling and related sensitivity to imperfections for the pillars similar to equally stable ones.

The model proposed in the work by Neut [Neut, 1969] served as a convenient object for studying the problems of related buckling of real structures. It was used by [Thompson & Lewis, 1972], [Neut, 1973], [Gilbert & Calladine, 1974], [Svensson & Croll, 1977] and others. It was concluded that just a related buckling in the elastic-plastic area usually determines the bearing
capacity of the pillar. In the paper of Thomson and Lewis [Thompson, Lewis, 1972], which used the Neut model in solving the problem of optimal design, it was shown that in the presence of local imperfections the optimal pillar is not equally stable and meets the conditions of high margin under the local form.

10.3. Systems of thin-walled bars

As a rule, thin-walled bars do not work singly, but as components of the flat or spatial systems. In this connection there appeared works, where the authors made an attempt to solve the relevant problems. The main problem here was the question of strain compatibility at the nodes of compression of thin-walled elements. This problem is easily solved for a continuous thin-walled bar, in which deplanations of end sections of the compressed elements coincide on intermediate support.

Perhaps the first who attempted to resolve the problem of analysis of the frame consisting of thin-walled bars was B.M. Gorbunov, who indicated the method of calculation of this type plane frames under spatial load [Gorbunov, 1943]. Here, as well as in their future works, B.M.Gorbunov and A.I.Strelbitskaya [Gorbunov, Strelbytskaya, 1948, 1950], which highlighted the issue of calculating thin-walled car frames, used the hypothesis of absolute rigidity of the welded node plate in its plane, that ensured the equality of deplanations of end sections of all rods that converged at a node (Fig. 10.8).

![Fig. 10.8. The node of car frame](image)

Many attempts were made to construct a sufficiently universal algorithm for calculating arbitrary thin-walled bar systems, but the
main problem was the formulation of boundary conditions at the ends of a thin-walled bar.

Some authors proceeded from the fact that deplanation is either absent or does not meet any obstacles at the end of the rod. One of the first in this direction was the work by Stavraky [Stavraky, 1948]. He considered the spatial (especially cyclically symmetric) thin-walled bar systems in the assumption that their nodes are either infinitely rigid and deplanation of end sections of all rods, converging in a node, is equal to zero, or the node structure is such that the deplanation is free for all end-sections.

Other studies used the hypothesis of deplanation equality at the ends of thin-walled bars converging in a node (see., e.g. [Urban, 1955]). This hypothesis is present in the work [Tusyn, 2009] in a slightly modified form, which introduced the concept of "deplanation conversion factor". These factors are rather set for transitions of "bar-bar" type than "bar - node" type and determined for a couple of bars adjacent to the node. It is unknown what will it be for the third, fourth, etc. bar, that can converge in a node. Statements like "deplanation axis orientation" and all other manipulations with the change in orientation of such "axes", that fill this book, is completely meaningless since deplanation is scalar.

In general, the incapability of the hypothesis of deplanation equality at all end sections adjacent to the node, was demonstrated in the work [Perelmuter, Yurchenko, 2012]. It was shown on simple examples that end sections deplanation of all the elements converging in a node, are not the same, and their value depends on the structure of the node, which strain takes a noticeable effect on the structure behaviour.

The concept of node superelement, introduced and studied in [Szymczak et al., 2003], [Mikulski, 2010], made it possible to take into account the interaction between internal forces and node deformation and end sections of thin-walled rods adjacent to the node. There have been proposed other methods of calculation that
allow one to account the deformability of nodal joints [Black, 1996], [Cichoń, Koczubiej, 2008], [Koczubiej, 2011].

References

Belyaev, V.N. (1932), “On design of spatial box-like system under the effect of twisting forces”, Tekhnika vozdushnogo flota, no. 4.
Bolotin, V.V. (1965), O ponyatii ustoichivosti v stroitelnoi mekhanike [On stability concept in structural mechanics], Stroyizdat, Moscow, USSR.
Vlasov, V.Z. (1940), Tonkostennye uprugie stershni [Thin-walled elastic bars], Gosstroyizdat, Moscow, USSR.
Gorbunov, B.N. and Strelbitskaya, A.I. (1948), Teoriya ram iztonkostennykh sterzhnei [Theory of frames of thinwalled bars], Gostekhizdat, Moscow, USSR.
Gorbunov, B.N. and Strelbitskaya, A.I. (1950), “Strength analysis of thin-walled bar structures”, Analysis of spatial structures, Iss. 1, pp. 97-162, Izdatelstvo ministerstva stroitelstva predpriyaniy mashinostroyeniya, Moscow, USSR.
Dzhanelidze, G.Yu. and Panovko, Ya.G. (1948), Statika uprugikh tonkostennykh sterzhnei otkrytogo profilya [Statics of elastic thin-walled bars], Goatekhizdat, Moscow, USSR.

Rzhansitsyn, A.R. (1941), “Complex resistance of thin-walled bars with undeformable loop within and outside the limits of elasticity”, Trudy laboratorii stroitelnoi mekaniki CNIPS, Stroyizdat, Moscow, USSR.

Rzhansitsyn, A.R. (1946), Raschet metallicheskih dvutavroveykh balok, poluchivshikh nachalnoe iskrivlenie v gorizontalnoi ploskosti [Calculation of metal H-beams initially curved in horizontal plane].

Slivker, V.I. (2005), Stroitelnaya mehanika Variatsionnuye osnovy [Structural mechanicica. Variational principles], Izdatelstvo ACB, Moscow, Russia.

Stavraki, L.N. (1948), Prostranstvennie pryamougolnye ramy iz tonkostennikh stertshnei [Spatial rectangular frames of thin-walled bars] (Chapter V in: Vainberg D.V. and Chudnovsky, V.G. Spatial frame skeletons of engineering structures), Gostekhzdat Ukrainy, Kiev, USSR.


Timoshenko, S.P. (2014), Vospominaniya [Memoirs], Vuzovskaya kniga, Moscow, Russia

Tusin, A.P. (2009), Chislennyi raschet konstruktsiy iz tonkostennikh stertshnei otkrytogo profilya [Numerical analysis of structures of thin-walled bars of open profile], Izdatelstvo ACB, Moscow, Russia.

Umansky, A.A. (1939), Kruchenie i izgib tonkostennikh aviakonstruktsiy [Torsion and bending of thin-walled aircraft structures], Oborongiz, Moscow, USSR


Urban, I.V. (1955), Teoriya raschota stertshnevlykh tonkostennikh konstruktsiy [Theory of design of the thin-walled bar structures], Transzheldorizdat, Moscow, USSR.

539


Eggenschwyler, A. (1921), Über die Drehungsbeanspruchung von dunndwandigen symmetrischen-formigen Querschnitten, Der Eisenbau, Vol. 12, no. 9, pp. 207-215.


541
Ostenfeld, A. (1931), Politteknisk Laezean stats Laboratorum for Bygningsstatik, Kopenhagen: Meddelelse.
Essay 11

FORMATION AND DEVELOPMENT OF THE NOTION OF THE STRUCTURE DESIGN MODEL
Only if we imagine too clearly various aspects of problems origin, it is hoped that we will choose rational mathematical models and use meaningful mathematical methods. As we will further repeatedly stress, the ideas play the same important part as equations, and creation and interpretation of mathematical models are even more important than those partial equations to which they are reduced.

R. Bellman

The creation of design models of structures is simultaneously a task of experts in structural mechanics and of those in structures. Various approximations of the real physical service of a structure may be created only under their joint work.

I.I. Goldenblat, V.L. Bazhenov
A design model reflects a designer’s idea of the real investigation object and peculiarities of its behavior. It is a simplified object model deprived of insignificant details and closely related to a set of some physical notions of laws, which control the investigation object behavior. Nowadays great experience exists in the design model development, and, preceding from this experience, in each specific case the following “type members” are used: such as shape idealization (a bar, plate, shell), regularities of material behavior (elastic, plastic, etc.), rule of these members coupling, etc. This designer’s arsenal was developed in the course of the whole history of structural mechanics as science and continues perfecting in the present.

11.1. Beginning of the path. Analysis of certain problems

An idea of a design model probably appeared simultaneously with science of strength in 1638, when the book by Galilei Disputes and Mathematical Proofs Concerning Two New Sciences was published, though the term design model appeared much later.

Just first attempts of design analysis of structure behavior, the attempts, which were aimed at the search of failure load, proceeded from certain hypotheses on location of dangerous section and force distribution in it. A set of these hypotheses could be called now a design model or design diagram.

Galilei thought rigid bodies to be inelastic and studied the problem on the bar strength, considering it in the state of failure (limit state in terms of the present). He attributed failure to two types of deformation – tension and bending.
In the first case strength was taken as proportional to the cross-section area, Galilei bound the second case with the first one, supposing that the cantilever break occurs by crack opening displacement from above and rotation about the lower rib, the whole section being uniformly extended (Fig. 11.1,a). A question of the break place was not raised in the explicite form, Galilei probably thought it obvious.

Several laws of stress distribution throughout the section height were further offered: Mariotte [Mariotte, 1686] and Leibniz [Leibniz, 1684] considered the distribution as linear with coordinate origin at the section edge (Fig. 11.1,b), while Parent [Parent, 1713] used the same law, but distributed the coordinate origin in the centre of the section height (Fig. 11.1, c). And only Navier [Navier, 1826] placed the coordinate origin in the centre of gravity (Fig. 11.1, d). At last Persi, Navier’s companion-in-arms at the school of bridges and roads, when developing Navier’s approach, introduced an idea of the section inertia moment, which has become and still remains a necessary attribute of the description of the schemes of bar structures.

The approach, which was based on the search for the break patterns and used a model in a form of a set of infinitely rigid blocks (the loss of link among them being connected to one or another extent with the break) prevailed in the problem of arch strain for a long period of time [Bershtein, 1936]. An important point is that the shearing schemes appeared among possible break schemes (Fig. 11.2).
The problem on the shape of flexible filament also belongs to the first period of formation of structural mechanics. It developed in two directions. The first one was connected with the problem formulated by Jacob Bernoulli: “to find what shape takes a rope freely hanged at two points”, and this direction has played the important part in formation of mathematical analysis.

Another direction may be connected with the name of Varignon, whose major work [Varignon, 1725] was published after his death and dedicated to the theory of funicular polygon – the design model, which is one of the principles of graphostatics. The problem on the funicular polygon attracted interest 100 years later in connection with the problem of design of suspended bridges, which chains were funicular polygons.

The discovery of Hooke’s law in 1660 and the establishing of Navier’s equations in 1821 are undoubtedly two important milestones in the further development of the theory initiated by Galilei. Hooke’s law has given a necessary experimental substantiation of the theory.

In the period between deducing the Hooke’s law and establishing general equations of elasticity theory obtained by Navier the interest of researchers was directed to solution and generalization of Galilei’s problem, to allied problems which concerned vibrations of bars and
plates. The first significant research in this field was made in 1705 by Jacob Bernoulli. It concerns the shape of an elastic curve of a bar and is based on the admission that the resistance of a bent bar depends on tension and compression of its longitudinal fibers.

When deriving the bar bending equation J. Bernoulli used the Hooke’s law, and besides, two following hypotheses:

- the sections, plane and perpendicular to the prism ribs before its bending, remain after the bending plane and normal to these ribs and fibers or longitudinal members that become curvilinear;
- the fibers, some of them being extended others – shortened, resist independently the bending, as if they were small isolated prisms, taking no effect on one another;

The same propositions were further taken by Euler in his research, which concerned the problems of elastic line and vibrations of thin bars. The Euler-Bernoulli design model of a bar presented an elastic bar in a form of a linear set of particles resisting the bending.

The successful development of the theory of thin bars, based on special hypotheses, has led to a conclusion that the theory of plates and shells may be constructed in the same way. Euler was the first, who was concerned with this problem. He offered to consider a bell as a set of thin rings, each of them behaving as a curved bar. This work was followed by the research of Jacob Bernoulli (junior). He considered a shell as a double layer of curved bars, the bars of one system intersecting with the bars of the other system at the right angle [Bernoulli, 1789]. Reducing a shell to a plane plate, he obtained an equation, which, as we know it today, was incorrect (he had not allowed for the bar twisting).

The attempt of Jacob Bernoulli was, probably, a purpose to obtain theoretical substantiation of experimental results by Chladni [Chladni, 1802], as to the figures of nodal lines observed under plate vibrations.
These results remained unexplained, when in 1809 the French Institute offered the problem on the tones of plate vibrations as a bonus theme of scientific work. After some attempts there appeared a work by Sophi Germain awarded in 1815 and published only six years later [Germain, 1821].

But a distinct design model of a bending plate was proposed only in 1850 by Kirchhoff [Kirchhoff, 1850], who based his theory of plates on the following two hypotheses, generally recognized nowadays:

- each straight line, which was first perpendicular to the midplane of a plate, remains under bending a straight line normal to the middle surface of a bent plane;
- elements of the plate midplane are not elongated at small plate deflections under transversal load.

These admissions are close by content to hypothesis of plane sections taken today in the elementary theory of bar bending.

### 11.2. Elastic bar systems

Before the 30’s of the 19th century structural mechanics had in possession the design models of bars, arches and plates – the base elements composing real structures. All these design models were realized separately, while they interact in numerous cases, being separate fragments of a more complex structure. If in the 18th century the design and technical development of civil engineering was concentrated on stone arches, in the 19th century the interest of engineers changed and they were oriented to analysis of skeleton constructions. In connection with a rapid development of railway engineering the transition from elementary carrying systems to composed constructive designs was much more prompt than in cast-in-place constructions (such as masonry and concrete), under these
conditions geometrical and physical properties of such structures became a logic abstraction of the design model in a form of a truss.

Fig. 11.3. Designs of Palladio bridges

The design of a girder composed of parallel chords linked by a lattice was first proposed by Palladio even in the 16th century (Fig. 11.3); but they became the important object of design analysis only in 1820 with the appearance of a girder structure with multilattice filling proposed by Town (Fig. 11.4).

Such girders were first considered as beams with a through wall. In Navier’s lectures they were presented in such a way [Navier, 1833/1878]. Navier used in his design only cross-sections of the top and bottom chords; he considered that it can be made, if a number of transversal members and “St. Andrew crosses”, the elements which intersect, are adjoined to the chords.

Such a design model was used to construct bridges with various structural systems, proposed by James Warren, Stephen Harriman Long, William Howe and other inventors [Perelmuter, 2015].

The considerable number of wooden bridges in North America was described by Carl Culmann, indicating those with the signs of

Karl Culmann
(1821 - 1881)
damage and failure, in spite of the generous use of materials [Culmann, 1851]. Culmann indicated different structural systems in those bridges and noticed that they could better perform their function on condition of correct design.

He has created a theory of braced structure, based on the following admissions:

- a system of filling with bars between the top and bottom chords should be made in such a way that all the bars formed triangles;
- the bars should have a possibility to turn in joints without restriction

Using the equilibrium conditions Culmann could calculate forces in the elements of any statically determined girder structure of the above type.

The work by Schwedler [Schwedler, 1851] appeared almost simultaneously, the author indicated (see Fig. 11.5):

“If a structure as a whole is considered as rigid, small resistances caused by elastic bending at the points a, d, c, etc. are insignificant compared with resistance of braces, or, in other words, it may be taken that separate components of the truss can turn at the points a, d, c, etc.”

Fig. 11.5. Hinged model of Schwedler frame

Schwedler has first performed the process of abstraction, which is typical of the structural theory: from the physical carrying structure (real through system, e.g. wooden truss) through the abstract carrying system (model of a through structure or girder bar system according to Culmann) to design model (of a hinged truss), described with the help of physico-geometrical properties.

The invention of design model in a form of hinged truss has become a key concept for development of the structural theory in the
second half of the 19th century. It is important that independent analysis of topologic structure might be made for this model. Such analysis was intensively developed in the works devoted to revealing kinematic properties of truss structures and then of the bar structures of any other kind.

Another important achievement, which originated from the design model of the truss, was the development of the conception of a node – a hinge joining the truss bars. Then the node hinge was considered as a material point, the equilibrium equation being formulated for it, and in this quality the node became an integral part of design model of the bar (and not only) systems.

The design model of a truss was rather evident, and the truss work proved very positive. Many engineers tried to construct the truss model. Schwedler developed hinged nodes for the bridge over the Brache river (now the Brda river) near Czersk built by his project in 1861 (Fig. 11.6, a).

But nine years later the other bridge over the Brache river was constructed near Bromberg (today Bydgoczsz), and now by Schwedler’s project using riveted joints (Fig. 11.6, b).

Fig. 11.6. Bridge truss nodes
Emil Winkler realized that the hinged model of the truss did not correspond to real work of metal trusses with riveted joint [Winkler, 1872]. Some bending moments appear due to the node rigidity in the truss bars, and as a result – additional stresses. The problem of their determining that first of all attracted attention of Manderla [Manderla, 1880], resulted in the appearance of the method of displacements.

Design models of skeleton structures, where most nodal joints are rigid, began quickly extending in connection with construction of reinforced concrete frames. And design model of a truss with ideal hinges proved to be a certain approximation to reality.

E.O. Paton has estimated the degree of relative increment of stresses that appear at the expense of node rigidity [Patton, 1901], and as his studies have shown, the more precise is the “truss approximation” the more is the flexibility of the truss bar elements. In essence, there arised an important question, which soon arised in other situations: the question on usability limits of one or another design model, of the necessity of its specification or cardinal change, when its parameters are outside a certain limit.

For example, S.P. Timoshenko has proposed a model of a bending beam, which difference from the Euler-Bernoulli model is that under deformation the cross-sections remain plane but not perpendicular to deformable midline of the bar, and inertial components connected with a turn of cross-sections are accounted in dynamics [Timoshenko, 1916]. E. Reissner has proposed an analogous perfection of the Kirchhoff model for plates [Reissner, 1945]. In both cases the point was in the necessity of introducing some specifications, when shearing deformations begin playing a considerable part.

The transition from analysis of truss models to investigation of frame systems became a large-scale one at the end of the 19th century
and especially in the first half of the 20th century. That was caused by the intensive use of cast-in-place reinforced concrete constructions in civil engineering.

Formulation of the problem on general rules of development of design models belongs to the first quarter of the 20th century, and here we should note the work of N. Gersevanov [Gersevanov, 1923], who has first formulated that:

- the design model is constructed proceeding from the expected form of failure and deformation based on the experience of building practices;
- the design model uses only hypotheses concerning the structure properties and actual loads, which allow developing the efficient methods of calculation.

The design model substantiation problem itself, besides the use of the results of experimental studies, developed in the following direction. Researchers proposed the ways to transform the design model of a more general form, e.g. a model of three-dimensional continuum of the problem of elasticity theory, to one or another model of the structural unit of a certain type. Such an approach was especially often used in development of the theory of plates and shells.

The first attempt of deriving equations of the theory of shells from the equations of elasticity theory was made by G. Aron [Aron, 1874]. Then this trend was developed in the works by A. Love [Love, 1888], A. Basset [Basset, 1892], H. Lamb [Lamb, 1890], A.I. Lurie [Lurie, 1947], et al.

After deriving the resolving equations of the shell theory and developing the corresponding design models researchers began developing various non-classical variants of the theory. Here one should recall the theory of shells of Timishenko-Reissner type that
allow for longitudinal shear deformations. Besides, the theory of the ribbed and multilayered shells may be referred to non-classical ones.

First works in this field for the plates reinforced by ribs were made by I.G. Bubnov [Bubnov, 1904]. The theory of ribbed shells of a general form was presented in the works by A.I. Lurie [Lurie, 1947] and V.Z. Vlasov [Vlasov, 1949]. A.I. Lurie considered ribs as the Kirchhoff-Clebsch bars, while V.Z. Vlasov considered them as thin-walled bars.

Multilayered shells were investigated from different viewpoints in a lot of works mainly in two basic directions. The first one includes theories based on design models, where kinematic hypotheses were taken for the whole set of layers. The first-stage researches have demonstrated nonperceptibility of this approach, if properties of the layers are essentially different; that is why the works of another direction were developed in recent years; a complicated design model, where kinematic hypotheses are taken separately for each layer, was used in these works.

11.3. Elastic foundation

When creating a design model, the research object is isolated from the environment; interaction with this object is realized in the form of loadings and effects, as well as fixation conditions. It is natural that the boundary between the design object and surrounding is conventional, and its choice affects essentially the design model construction.

It is customary to provide a design model with a motionless undeformable element, which is conventionally called “the earth”; the reference system is usually connected with this element, and equilibrium conditions should not be met for the latter. All conditions of support were formulated as a rigid “adjustment to earth” over many years. This tradition was first broken by Emile Winkler [Winkler, 1867], who has considered a problem on a beam on elastic foundation in his lecture on elasticity and strength. This problem has
arisen in connection with the analysis of the railway rail path, and Winkler proceeded from the fact that each sleeper subsides under load by the value determined by foundation $C$ rigidity, independent of behavior of the next non-loaded sleeper. A mechanical model of this type is presented in Fig. 11.7.

But most real soils are characterized by distribution capacity, when, in contrast to the Winkler design model, not only directly loaded parts of foundation but also the adjacent areas of non-loaded soil are involved in the work. A model in a form of elastic half-space was proposes as alternative [Wieghart, 1922]; it presents the foundation as isotropic elastic body of infinite sizes in plan and depth, and a series of other models with two bed coefficients, which remove the main defect of Winkler’s model (allow accounting soil distributing capacity). The same distribution capacity is characteristic of two-parameter design models, which do not almost complicate mathematical posing of the problem compared with Winkler’s model.

![Fig. 11.7. Winkler’s design diagram](image)

According to M.M. Filonenko-Borodych (Fig. 11.8) the two-parameter model of the elastic foundation is a bilaterally unconstrained unextensible filament, which is stretched with force $C_2$ and connects upper ends of continuously arranged springs with distributed rigidity [Filonenko-Borodych, 1945].

A discrete two-parameter model of P.L. Pasternak is shown on Fig. 11.9. In this model $z$-like absolutely rigid elements are connected with the earth by a set of springs, which are discrete analogue of coefficient $C_l$ (which characterizes the foundation rigidity under compression), while spring arranged between neighboring $z$-like elements serve a discrete analogue of coefficient

556
$C_2$, which characterizes the foundation shearing rigidity [Pasternak, 1954].

![Diagram of M.M. Filonenko-Borodych's design model](image)

**Fig. 11.8.** Design model of M.M. Filonenko-Borodych

Today, when the method of object design by the scheme *structure-foundation* has become a practical standard, the number of the used design models of the foundation has increased to very large values. But a conventional boundary, which distinguishes the studied object in the environment, has essentially extended.

![Diagram of P.L. Pasternak's design model](image)

**Fig. 11.9.** Design model of P.L. Pasternak

![Diagram of a building with foundation](image)

**Fig. 11.10.** Design model of a building with foundation
A rather typical example is shown in Fig. 11.10, where the number of the degrees of freedom of the overground part of the building is 96720, and when considering joint work of the building and the foundation it becomes 305837.

The introduction in practice of analysis of the model of elastic foundation has broken the tradition of using the black-and-white logic, when analyzing constraints imposed on the system, (the constraint is present – the constraint is absent). This circumstance has led to the idea of abandoning the idealization of the properties of nodal joints and of using a model of compliant nodes in the design diagrams. In the present this approach is regulated by some normative documents. For example, Eurocode-3 uses such notions as a rigid node, half-rigid node and compliant node.

Besides, the allowance for the elastic foundation has broken one more stable principle that a concrete element of the model corresponds to a concrete element of the physical system. It may be said that the whole model is physical, if all its elements have concrete physical prototypes. But in this case there exists a model element, which has no physical prototype – the half-infinite parts of the foundation are modeled by the finite-size mechanical elements (springs fixed in the nodes, which have no analogue in the physical system). One can say that in such a case we rather model a function than a geometrical image.

11.4. Structural analysis

A detailed analysis of separate problems and simple objects has led to development of such a notion as a material point, absolutely solid body, elastic bar, plate, etc. The properties were studied for them, which are used, when a more complex model of the problem required is constructed using such parts as of a certain “constructor”. And a problem of analysis of the composed design model of such kind appeared in natural way. Most researches were devoted to design models of trusses; they attracted by their “homogeneity” and
distinct division of topologic (structure, fixing) and metric (node coordinates, section sizes) data.

As to the complete system of determining equations we should say that Alfred Clebsch has shown in his work [Clebsch, 1862] that a set of equilibrium equations and those of deformation compatibility for an arbitrary truss has the solution [Clebsch, 1883]. But the problem of solution possibility was, first of all, considered from the viewpoint of equilibrium equations issuing from the composed design model – equations of analysis of its statical determinability and invariability.

Even in 1837 A.F. Möbius proved the theorem that to obtain a rigid invariable structure in a truss with \( n \) hinges it is necessary to have no less than \( 2n-3 \) bars in a plane system and no less than \( 3n-6 \) bars in the case of a spatial system [Möbius 1837]. In so doing he has probably first indicated a possibility of existence of exceptional configurations, when one observes infinitesimal mobility without bar deformation (a case of instant variability in the current terms).

When studying these cases Möbius has found that therewith a determinant of the set of equilibrium equations becomes zero. The connection between the variability criterion and degeneracy of the system of resolving equations became after a while the basis for computer analysis of kinematic properties of design system of any (not only truss) type. The results obtained by A.F. Möbius, which then remained unknown, were found again by P.L. Chebyshev [Chebyshev, 1870] and Otto Mohr [Mohr, 1874] and only then entered in the design practice.

Mass enthusiasm as to the method of forces, characteristic of the end of the 19th and first half of the 20th century, resulted in the appearance of various procedures of construction of the basic system.
of this method and in the problems of revealing the redundant constraints in statically undeterminable design models. Relations between the properties of static determinability, invariability and ability to realize the pre-stress were studied in detail for the bar systems. Researchers indicated methods for determining statical-kinematic properties based on reducing the system to a certain number of hard discs, bound by bars-restraints. They also introduced the notions of simple and multiple hinges and other idealized elements of the design model.

Noticeable changes in the concepts of design model are connected with the transition to displacement method analysis. In the displacement method the system elements are considered to be connected with the nodes of the design model, they are not connected directly with one another. The above peculiarity of the design model construction was often disregarded by engineers educated on the ideas of design model in the style of the force method, it is not always seen, when using the methods of design model representation that is traditional for the force method.

Thus, the design system presented in Fig. 11.11, a in traditional form, inherent in the force method, can suggest the point-to-point connection of elements with one another, while a more detailed representation in Fig. 11.11, b allows avoiding such conclusion. Note
also that in the detailed presentation one can also see other peculiarities of the design model implementation, in particular, a possibility to meet similar kinematic conditions with using various sets of constraints imposed on the nodes, and conditions of elements connection with the nodes.

Neglect of the above difference is not always safe. For example, from the viewpoint of kinematic properties of the problem two variants of the design model, presented in Fig. 11.12, have equal rights (a beam is fastened in its left end and hinged in the right one).

But from the viewpoint of giving forces these variants differ – in the scheme of Fig. 11.12, b the moment is transferred to the bar, and node 2 in this scheme will turn, and in the scheme of Fig. 11.12, a the moment is not transferred, and node 2 of this scheme will have a zero turning angle. For a bending moment to appear in a bar in the scheme on Fig. 11.12, a, it should not be considered as nodal, but applied to the bar in the section near a node.

![Fig. 11.12. Two variants of presentation of one design model](image)

The above division of topologic and metric properties of the design model gave impetus to the works connected with the use of graph theory to analyze properties of bar systems. Such was the approach in the pioneer publications [Fenves & Branim, 1963], [Perelmuter, 1965], while in the work by Di Mattio [Di Mattio, 1963] the degree of static indeterminability is studied just as topologic property of the design system. Later on there appeared works, where topologic connectedness of design model was compared with the structure of distribution of nonzero elements of rigidity matrix of the system that is analyzed, and a
possibility of optimal enumeration of unknowns [Akyjz & Utku, 1968], [Clempert, 1973]. Researchers proposed some artificial techniques aimed at the improvement of the above structure even at the expense of increasing the rigidity matrix order [Perelmuter, Slivker, 1976].

Practical interest to the analysis of suspended and rope systems, characteristic of the works of the second half of the 20th century, resulted in a detailed study of topologic and metric properties of degenerated (instantly variable and instantly rigid) systems. There appeared a number of fundamental works [Kuznetsov, 1960], a lot of researches were initiated by introduction of the systems of “tensegrity” type by Buckminster Fuller [Fuller, 1961]. He used this term to indicate the frame structures, involving continuous chains of members, which work in tension, and inserted members, which work in compression.

Study of properties of instantly rigid systems and systems of “tensegrity” type preferred to come back to general principles of analysis of static-kinematic properties of the composite design models [Shulkin, 1977], [Calladine, 1978], [Connelly, 1980]. The systems with unilateral constraints, possible combinations of static and kinematic properties being established for them [Perelmuter, 1968], were also considered.

Fig. 11.13. Examples of tensegrity structures
11.5. Design models of the finite element method

The appearance and development of the finite element method (FEM) has told essentially on the problem of choice and substantiation of the design model. Even a description of geometrical pattern of the structure has become a choice of a designer, as it occurs in the problems, where a curved shell surface is modeled by a multiface set of plane finite elements. However, such a problem also appeared before, when they used an approximate description of curved bars by a certain polygon.

A possibility to present a design model as a set of finite elements, their quantity in configurations being limited by nothing but the library of finite elements at designer’s disposal, has raised in a new fashion the question on the number of basic unknowns, degree of kinematic and static indeterminability and other still inviolable characteristics of the design model of the building. The number of unknown displacements (the degree of kinematic indeterminability) stopped being the problem feature and became a subject of designer’s self-will.

Some “standard” approaches to composing the finite element design models were developed for most types of structural systems. For example, at the first stage of using FEM a design model of a thin-walled fuselage structures and aircraft wings became popular; it was composed of shearing panels and a frame of bars, supporting them at the edges, able to take up only longitudinal forces.

In such wing model (Fig. 11.14) the bars simulate the work of longitudinal members of the wing structure under load; these members are subject to compression and extension under the wing bending. Plates simulate the work of walls, which prevent shearing, as well as the external and internal wing covering.

This model was propagated by G. Argyris [Argyris & Kelsey, 1954], and though it appeared long before FEM [Ebner & Köller, 1937], [Umansky, 1950], its implementation proved acceptable only in the framework of FME, though it existed there just for several years. Potentialities of computation complexes, which were quickly perfected,
permitted even in the 70’s of the 20th century specifying the design model and taking account of bending strains of the supporting frame.

![Fig. 11.14. A thin-wall system model](image)

There originated new approaches to design model development for the plates and shells with ribs. There also appeared competing propositions as well as the problem of their verification. The corresponding example of the variant of a design model composed of solid elements (Fig. 11.15, b) and the variant of modeling using plate and bar elements connected through infinitely rigid inserts (Fig. 11.15, c) are presented in Fig. 11.15.

![Fig. 11.15. Possible variants of a ribbed plate modeling](image)

In industrial and civil buildings, where, e.g. the through columns are used, designers often abandon presentation of such a column as a solid rigid bar, but use a more detailed description of the structure. And above all, they have practically given up plane design models and
resolve all problems with the use of three-dimensional models. The main trend of development is now the use of more oversized design models; the number of unknowns of the order of hundreds of thousands became ordinary in the design practice; in so doing such an extensive detailing is not always necessary but connected with formal construction of the design model by the data of graphical program used for making drawings.

A possibility of modeling structures of arbitrary nature, including those which separate parts are presented by bars, others – by plates or shells, the third – by three-dimensional bodies, raised questions as to connection of finite elements of different type and arising problems, caused by the difference of nodal degrees of freedom in different type elements. A necessity of using special techniques, for example, such as the introduction of the bar element into the rigid disc body, has been indicated [Perelmutter, Slivker, 2001].

The finite element method is realized in program complexes with increasing potentialities. This fact often serves as a basis for unbounded increase of design models, realize under the slogan of specifying. Excessive detailing of the system is often a result of designer’s reaction to a necessity of revision of the data from extremal results in the absence of previous information on the place of this result origin. Then, to make sure, a detailed design model is used, which will probably allow one to notice a necessary result.

Fig. 11.16. Modeling of interconnection of a bar and disc: a – task; b – variant, which does not provide a restraint; c – recommended decision

But it should be allowed for that the above result may be lost because of difficulties in analysis and comprehension of excessive
information. And this so-called revision often leads to shading the basic features of the structure service, and its simplified variant is considered for their analysis in parallel with a detailed design model. In so doing there sometimes arises a nontrivial problem of results comparison; it is especially difficult, when there is no exact fit between the elements of compared design models [Perelmuter, Slivker, 2001].

And finally note that the analysis of design models of the finite element method is closely connected with the problem of method convergence. Mathematical proofs of the corresponding facts (for example, a demand of compatibility of the fields of displacements) require their interpretation in terms of design models, which are successively “concentrated”. In particular, in the case of incompatible elements it should be remembered that the solving of design problem is equivalent to minimization of full potential energy of the system (Lagrange functional), and approximation of the displacement field by a certain finite set of preset functions narrows a possibility of arbitrary deformation, that is it may be treated as the imposing of some constraints. If elements are incompatible, some displacements are possible at their boundaries; these displacements do not exist in the continual design model (for example, mutual rotation angles of plates), and correspond to the absence of some constraints.

When the number of finite elements increases and their sizes decrease, the total number of the structure degrees of freedom grows, and thus, the effect of the imposed nodal constraints is reduced. This process, certain conditions being fulfilled, provides the method convergence for compatible finite elements. On the other hand, the same process leads to the decrease of mutual displacements at interelement boundaries in incompatible elements that may be treated as a certain locking of the preliminarily left constraints. Thus the convergence of incompatible elements can take place only when positive tendencies of overcoming the imposed constraints prevail over this negative tendency of imposing the constraints at interelement boundaries.
Other approximations are sometimes realized simultaneously with approximation of the displacement field; those are connected with a necessity of using a finite-element model that is the substitution of the structure geometry by that similar to it. In the system geometry approximation, both the geometry and boundary conditions may change, since the latter belong now to the boundaries with other configuration. Here one can run across the reefs, since the passage to the limit of the outline form is not necessarily accompanied by passage to the limit of kinematic properties. That is evidenced by well-known Saponzhyan paradox for a freely supported polygonal plate [Panovko, 1985].

At the outset of FEM use they discussed a so-called problem of “small length” of the bar finite element, when a stress was made on the fact that a bar was defined (in courses of material resistance or structural mechanics) as the object, which one size (length) exceeded considerably other ones that defined cross-section dimensions. But in the design model in use the bar as the design model element may be found to be very short. There seemingly appears a violence of agreements concerning a bar definition. In fact, there is no violation, since the admission on a sufficient length of the bar was only required to substantiate the type of the corresponding differential equation. As to the method of its solution, when a bar is divided into rather small parts (read interval of integration), this has no effect on the equation form.

**11.6. Some new tendencies**

In recent decades there appeared a branch of structural mechanics based on probabilistic analysis, which started its intensive development. The approaches to the design system itself and to development of the corresponding design model have undergone considerable changes.

The whole identity of parameters of all similar structure elements is foreseen in a determined variant. It is considered that
all the headers of a three-dimensional framework have equal spans, all the columns of these headers – similar sections, etc. In so doing all such type elements are reduced to one representative, and sometimes to its one section. Such an approach is acceptable under the approach to design contained in the design norms. The approach is based on the half-probabilistic method of limiting states. Then all the probabilistic characteristics are formulated and estimated beyond the design model, and a design contains only some guaranteed the worst statistical estimates of means, standards, quantiles, etc., which are really the same for all identical elements, since one of definitions of their “identity” is the identity of the distribution law.

The transition to really probabilistic design was connected with the fact that one has not to operate on distribution parameters of random values but on distributions themselves, when random parameters operate not outside but inside the design model. And in such a model each of “identical” substructures should be defined in a form of a set of mutually independent (or correlated) random values (probably functions) with the same distribution laws [Bolotin, 1971]. In such formulation one cannot imagine a design model, e.g. of a plane problem, which provides identity (or rigid correlation with correlation coefficient equal to one) of all plane subsystems distributed in parallel.

The following should be noted: a further detailing of the models, when passing to analysis of multielement structures (a building as a whole), requires the involvement of the great number of parameters, used for the model description. If these parameters are random values, which probabilistic properties are statistically justified, the degree of design model indeterminacy as a whole increases with the number of such parameters. Thus, if a certain result of analysis is in linear relationship with \( N \) independent random parameters (e.g. external loads in the system nodes), the standard of this result is proportional to that of the input data (here loads) with a multiplier of
\((N)^{1/2}\) order. It is simple to estimate what is the probability of the results of analysis at very high \(N\) values.

There are more detailed propositions as to determining the effect of output data accuracy on the results of analysis (see, for example, [Podolsky, 1984]. They evidence that the information on the input parameters being insufficient, it is expedient to use simple design models. Such a peculiarity of design modeling is connected with the fact that the loss of information because of incompleteness of output data can exceed information accumulation at the expense of refinement of the design model.

The foregoing should be never considered as a panegyric to “good old days”, when everything was solved using the formula \(qL^2/8\) and counted using a slide rule. The thoughtless complication of design models would be substituted by new culture of their use that includes also the estimation of possible uncertainty of solution. Now, having the modern means for analysis of complex and supercomplex systems, we study them in formulation of the problem, which rather corresponds to the 19th than to the 21st century.

At last, it should be mentioned that the above problem of output data influence on the design model form is inherent not only in the probabilistic problems. The choice and justification of the design model cannot be separated from the level of information as to the structure, which is designed as well as from the method of solving the mathematical problem, formed as a result of using the chosen design model.

And what is more, a lot of mathematical operations used, when solving a problem, often have a mechanical interpretation, which use helps understand the features of calculation process. As an example, we can refer to the interpretation of Gaussian algorithm for solving a canonical system of linear algebraic equations, as to the sequence of imposing (force method) or taking off (displacement method) constraints [Gantmacher, 1967]. Such illustration favors a better understanding of the problem.
References


Bolotin, V.V. (1971), Primennyem metodov teorii veroyatnosti i teorii nadyoznosti v raschototakh sooruzheniy [Use of the methods of elasticity theory in structures design], Stroyizdat, Moscow, USSR.

Bubnov, I.G. (1904), Napryazhenie v obshivke sudov ot davleniya vody [Stress in boat plating from water pressure], Izdatelstvo Politekhnicheskogo institute, StPetersburg, Russia.


Gantmacher, F.R. (1967), Teoriya matrits [Theory of matrices], Nauka, Moscow, USSR.

Gersevanov, N.M. (1923), Primennyemiatematicheskoi logiki k raschotu konstruktsiy [Application of mathematical logic to structure analysis], ONTI, Moscow, USSR.


Kuznetsov, E.N. (1960), Vvedeniye v teoriiy visyachikh system [Introduction in the theory of suspended systems], Stroyizdat, Moscow, USSR.

Lurie, A.I. (1947), Statika tonkostennykh uprugikh obolochek [Statics of thin-wall elastic shells], Gostekhizdat, Moscow, USSR.


Myshkis, A.D. (1994), Elementy teorii matematicheskikh modeleii [Elements of the theory of mathematical models], Fizmatlit, Moscow, Russia.

Pasternak, P.L. (1954), Osnovy novogo metoda raschota fundamentov na uprugom osnovanii pri pomoshchi dvukh koeffitsientov posteli [Principles of a new method of foundation design on elastic base with the help of two bed coefficients], Gosstroyizdat, Moscow-Leningrad, USSR.

Paton, E.O. (1901), Raschet skvaznykh ferm s zhdestkimi uzlami [Design of through trusses with rigid nodes], Moscow, Russia.


Perelmuter, A.V. (2015), Ocherki po istorii metallicheskih konstruktsiy [Essays in the history of metal structures], Izdatelstvo SCAD SOFT, Izdatselsky dom ACB, Moscow, Russia.


Perelmuter, A.V. and Slivker, V.I. (2001), Raschotnyie modeli sooruzheniy i vozmozhnost ih analiza [Design models of structures and possibility of their analysis], VVP Kompas, Kiev, Ukraine.


Chladni, E.F.F. (1802), Die Akustik, Leipzig, Germany.

Clebsch, A. Theorie der Elastizität fester Körper, 1883.


Möbius, A.F. (1837), Lehrbuch der Statik, Bahd 2.


Navier, C.L.M.H. (1826), Resume des lecons donnees a l'ecole des ponts et chaussees sur l'appliqiation de la mecanique a l'etablissement des constructions et des machines. Premiere Partie, Paris, France.


Varignon, P. (1725), Nouvelle mécanique ou statique, Paris, France.


573
Winkler, E. (1867), Die Lerne von der Elastizität und Festigkeit, Prag, Czeckia.
Winkler, E. (1872), Die Gitterträger und Lager gerader Träger eiserner Brücken, Carl Gerold's Sohn, Vienna, Austria.
## CONTENTS

Foreword.......................................................................................................................... 2

Essay 1. STRUCTURAL MECHANICS AND NOT ONLY: DEVELOPMENT OF REQUIREMENTS TO NO-FAILURE OF STRUCTURES.................................................................................. 9

Introduction ......................................................................................................................... 11

1.1. Prehistory .................................................................................................................... 12

1.2. First investigation ....................................................................................................... 15

1.3. Admissible stress ......................................................................................................... 17

1.4. Breaking load ............................................................................................................. 22

1.5. New ideas ................................................................................................................... 29

1.6. Use of reliability theory ............................................................................................. 37

1.7. Modeling of loadings .................................................................................................. 44

1.8. Optimal reliability level ............................................................................................ 47

1.9. Failure criteria ............................................................................................................ 51

1.10. Conventional and real safety factor ........................................................................ 56

1.11. The design calculation cannot allow for everything .............................................. 58

References ......................................................................................................................... 61

Essay 2. THE HISTORY OF THE CONCEPT OF STRESS .................................................. 69

Introduction ......................................................................................................................... 71

2.1. A. Cauchy generalized stress principle ..................................................................... 76

2.2. Main stages of the history of Cauchy’s generalized principle ................................. 67

2.2.1. Antiquity ................................................................................................................. 88

2.2.2. Renaissance. Leonardo da Vinci ....................................................................... 92

2.2.3. Galilei .................................................................................................................... 93

2.2.4. Hooke, Mariotte, Young ..................................................................................... 98


2.2.6. S. Stevin’s principle of solidification. Works in hydrostatics, hydraulics and hydrodynamics .................................................................................................................. 108

2.2.7. Bending of a beam ............................................................................................... 111

2.2.8. Determination of shearing stress. Coulomb ..................................................... 120

2.2.9. Allowance for shear under bending of beams. D.I. Zhuravsky .................... 125

References ......................................................................................................................... 128

Essay 3. STAGES OF DEVELOPMENT OF STATIC STABILITY THEORY ................................................................. 137

3.1. General principles and theorems of stability ............................................................. 139

3.2. Stability of compressed bars .................................................................................... 156
3.3. Eccentrically compressed bars ........................................ 164
3.4. Stability of a plane bending form, bending torsion form of buckling ........................................ 168
3.5. Stability of curvilinear bar ........................................ 170
3.6. Stability of plates ........................................ 172
3.7. Stability of shells ........................................ 175
3.8. Stability of multi-element elastic systems ........................................ 180
3.9. Search of critical load ........................................ 187
   3.9.1. Qualitative methods ........................................ 189
   3.9.2. Numerical methods in stability problems ........................................ 192
References ........................................ 193

Essay 4. APPEARANCE AND FORMATION OF THE CALCULUS OF VARIATIONS ........................................ 205
4.1. First variational problems ........................................ 207
4.2. Legendre transformation. Young inequality. Euler’s theorem on homogeneous functions ........................................ 225
4.3. Duality for variational principles ........................................ 229
References ........................................ 233

Essay 5. VARIATIONAL AND EXTREMAL PRINCIPLES OF MECHANICS. THE HISTORY OF DEVELOPMENT ........................................ 241
Introduction ........................................ 243
5.1. Principle of possible displacements ........................................ 248
5.2. Fundamentals of statics ........................................ 251
5.3. First variational principles. Kepller, Fermat, Principia by Newton. Leibniz, Maupertuis ........................................ 270
5.5. Dynamics. Principles of D’Alembert, Jordain, Gauss, Hertz ........................................ 284
5.6. The principle of Hamilton-Ostrogradsky. The dual principle of Hamilton-Poincaré ........................................ 288
References ........................................ 296

Essay 6. BASIC VARIATIONAL PRINCIPLES AND FUNCTIONALS OF STRUCTURAL MECHANICS ........................................ 309
6.1. Variational principles of mechanics of solids ........................................ 311
6.2. Principles of Lagrange and Castigliano ........................................ 330
6.3. Clapeyron theorem. Theorems that bind the volume and surface integrals. Integral formula. Papkovych formula ........................................ 341
6.4. Conclusions ........................................ 351
References ........................................ 357

576
Essay 7. DUAL NATURE OF THE PROBLEMS OF STRUCTURAL MECHANICS. ON THE HISTORY OF THE FORCE METHOD AND DISPLACEMENT METHOD ................................................................. 369
   Introduction ........................................................................................................ 371
   7.1. The forms of expression of potential energy. Partial derivatives of potential energy ................................................................. 376
   7.2. On the history of the force method and displacement method .......... 385
   7.3. Matrix formulation. Argyris ..................................................................... 396
   References ........................................................................................................... 403

Essay 8. FRAGMENTS OF THE HISTORY OF FINITE ELEMENT METHOD ................................................................. 413
   8.1. Prehistory .................................................................................................. 415
      8.1.1. Physical discretization ..................................................................... 415
      8.1.2. Displacement method .................................................................... 419
      8.1.3. Usage of localized functions .............................................................. 422
      8.1.4. Matrix formulation .......................................................................... 423
   8.2. Origin of the method .................................................................................. 426
      8.2.1. First steps ............................................................................................ 426
      8.2.2. Chain reaction ................................................................................... 430
      8.2.3. Extension to plates and shells .............................................................. 431
      8.2.4. Isoparametric element ...................................................................... 433
   8.3. Search of a rigorous justification ............................................................... 435
   8.4. Other variants of FEM .............................................................................. 437
      8.4.1. Ritz method and Galerkin’s method .................................................. 437
      8.4.2. The use of other functional ................................................................. 438
      8.4.3. Discrete-continual (semianalytic) FEM ............................................ 442
      8.4.4. Superelement approach .................................................................. 446
   8.5. Software implementations ......................................................................... 449
      8.5.1. Choosing of the method ..................................................................... 449
      8.5.2. Formation of software architecture .................................................. 450
      8.5.3. Search of solvers ................................................................................ 452
      8.5.3. Step-by-step procedure ..................................................................... 455
   References .......................................................................................................... 458

Essay 9. STAGES OF DEVELOPMENT OF THE PROBLEMS OF SYNTHESIS IN STRUCTURAL THEORY ................................................................. 469
   Introduction .................................................................................................... 471
   9.1. Inverse problem of structural mechanics ............................................... 472
   9.2. The beginning of the way ......................................................................... 473
   9.3. The origin of the theory .......................................................................... 479

577
9.3.1. Equal strength and method of defined stress .......................... 480
9.3.2. Energy approach .......................................................... 484
9.3.3. Optimization as a problem of mathematical programming ........ 487
9.4. Synthesis of the scheme ...................................................... 494
9.4.1. Continuous cross-system ............................................... 494
9.4.2. Using of special models of the finite element method .............. 497
9.5. Calculation focused on optimal design .................................. 502
9.6. Untraditional optimization problem ..................................... 503
References ..................................................................................... 506
Essay 10. STATICS AND STABILITY OF THIN-WALLED BARS............. 517
10.1. Thin-walled elastic bar ....................................................... 519
  10.1.1. Classical period .......................................................... 519
  10.1.2. Back to sources ......................................................... 529
10.2. Stability of equilibrium ..................................................... 531
  10.2.1. Action of longitudinal force ........................................ 531
  10.2.2. Local stability .......................................................... 532
10.3. Systems of thin-walled bars .............................................. 536
References ..................................................................................... 538
Essay 11. FORMATION AND DEVELOPMENT OF THE NOTION OF THE STRUCTURE DESIGN MODEL ............................................. 543
11.1. Beginning of the path. Analysis of certain problems ............... 545
11.2. Elastic bar systems ............................................................ 549
11.3. Elastic foundation ............................................................. 555
11.4. Structural analysis ............................................................ 558
11.5. Design models of the finite element method .......................... 563
11.6. Some new tendencies ....................................................... 567
References ..................................................................................... 570
Personalia

Bazhenov Viktor Andriyovych,
Doctor of science (Engineering), Professor, Academician of the National Academy of Pedagogical Sciences of Ukraine, Honored man in science and technology in Ukraine. Laureate of State Prizes of Ukraine in the field of Science and Technology, Education, Head of the Department of Structural Mechanics of the Kyiv National University of Construction and Architecture. Sphere of scientific interests: structural mechanics, theory and methods of numerical research of nonlinear deformation and fracture of inhomogeneous shell and massive structures.

Perelmutter Anatolii Viktorovych
Doctor of science (Engineering), Professor, Foreign member of Russian Academy of Architecture and Construction Sciences, chief research worker of State Enterprise SCAD Soft Ltd. Sphere of scientific interests: structural mechanics, building structures, reliability and safety, information technologies.

Vorona Yurii Volodymyrovych,
Candidate of science (Engineering), senior scientific worker, Professor of the Department of Structural Mechanics of the Kyiv National University of Construction and Architecture. Sphere of scientific interests: structural mechanics, application of the method of boundary integral equations to the problems of the theory of thermoelasticity and poroelasticity.
Buy your books fast and straightforward online - at one of the world’s fastest growing online book stores! Environmentally sound due to Print-on-Demand technologies.

Buy your books online at
www.get-morebooks.com

Kaufen Sie Ihre Bücher schnell und unkompliziert online – auf einer der am schnellsten wachsenden Buchhandelsplattformen weltweit!
Dank Print-On-Demand umwelt- und ressourcenschonend produziert.

Bücher schneller online kaufen
www.morebooks.de